ELECTRON DISTRIBUTION FUNCTION IN A NONEQUIPARTITION PLASMA

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ABSTRACT

Solutions to the kinetic equation of a steady, homogeneous plasma of arbitrary degree of ionization subjected to strong electric fields are developed. Nonelastic as well as elastic encounters are included in the analysis. Expressions for the current density, electrical conductivity and electron temperature of the plasma are also presented. Numerical results are illustrated for the case when the nonelastic effects are neglected. Calculations are presented showing the transition of the isotropic part f^O of the distribution function from a gas-temperature Maxwellian at near equilibrium conditions to an electron-temperature Maxwellian under nonequipartition conditions. It was found that whenever the electron-electron to electron-neutral collision-frequency ratio was much greater than the electron-to-heavy-particle mass ratio, f^O is Maxwellian. When f^O is Maxwellian, the Chapman-Enskog and Spitzer-Härm conductivity expressions developed for the case of weak electric fields are shown to be applicable for strong electric fields providing that in these expressions the electron temperature replaces the gas temperature. An approximate form for f^O suggested by Ginzburg is compared with the exact expression. The accuracy of the use of a Maxwellian rather than the nonequilibrium distribution function in the calculation of the electrical conductivity is assessed.

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NOMENCLATURE

This is a partial alphabetical listing of the present nomenclature. All symbols are defined locally in the text where they are first used. Some symbols whose use remains local will not be found on this list.

An attempt has been made to keep the nomenclature phenomenologically familiar. This effort has resulted in what may appear to be a double meaning for a few of the symbols. In any apparent conflict of definitions, local usage of the term should leave no doubt as to the intended meaning.

Latin Letters

A_s	Dimensionless field strength parameter. $A_s \equiv \frac{2\gamma}{\beta_e n_e \Gamma_{ee}}$
$A(v)$	Coefficient of isotropic distribution function. [See Eq. (53).]
$a(v), b(v), c(v)$	Coefficients of Eq. (52). [See Table I and Eqs. (48) and (51).]
$a'(v), b'(v), c'(v)$	Simplifications of $a(v)$, $b(v)$, and $c(v)$. [See Eqs. (65), (66), and (67).]
$B(v)$	Function defined by (63).
b_0	Impact parameter for a 90° Coulomb deflection. $b_0 \equiv \left(\frac{e^2}{u_1 v^2}\right)$
c	Speed of light.
$D(v), D(x)$	Function describing anisotropy of distribution function. [See Eq. (33) and Appendix B.]
$\vec{E}$, E	Electric field vector, magnitude.
E_c	Critical electric field for runaway. $E_c \equiv 0.427\, n_e(m_e/e)\Gamma_{ee}\beta_e$
e	Electron or absolute magnitude of its electrical charge.

F_{NE}	Nonelastic collisional "gain" term. F_{NE} is defined by Eq. (A-87).
$F_{I_{j\ell}}(v)$, $F_{R_{\ell j}}(v)$	Functions used to represent the contributions to F_{NE} which result from ionization and recombination. [See Appendix A.II.B.]
f_e, f_e', f_{e_m}	Electron velocity distribution functions. $f_e \equiv f_e(\vec{v},t)$, $f_e' \equiv f_e(\vec{v}',t)$, $f_{e_m} \equiv f_e(\vec{v}_m,t)$ (Subscript m defined below.)
f_h, f'_h	Velocity distribution functions for heavy particles. (Subscript h defined below.)
f^0, f^1	Isotropic and anisotropic parts of distribution function for electrons with speed v.
f_m^0, f_m^1	Isotropic and anisotropic parts of distribution function for electrons with speed v_m.
$\tilde{f}^0$	Electron temperature Maxwellian distribution function for electrons. $\tilde{f}^0 \equiv n_e\left(\frac{\beta_e}{\pi}\right)^{3/2} \exp(-\beta_e v^2)$
$\tilde{f}_T^0$	Gas temperature Maxwellian distribution function for electrons. $\tilde{f}_T^0 \equiv n_e\left(\frac{\beta}{\pi}\right)^{3/2} \exp(-\beta v^2)$
f^{01}	Approximate representation for f^0. [See Eq. (61).]
$\vec{G}$	Center-of-mass velocity. (See Appendix A.I.B.)
$G(v)$	Rosenbluth potential function associated with the diffusion coefficient of the Fokker-Planck equation. (Appendix A.I.)
g_h	Relative velocity defined by $g_h \equiv \lvert\vec{v} - \vec{v}_h\rvert$.
g_{mh}	Relative velocity defined by $g_{mh} \equiv \lvert\vec{v}_m - \vec{v}_h\rvert$.
$g^0(v)$	Function defined by Eqs. (62) and (64).
H	Parameter associated with Saha equation. $H \equiv \frac{h^3}{m_e^3}\frac{\omega_j}{2\omega_\ell}$

$H(v)$	Rosenbluth potential function associated with the friction coefficient of the Fokker-Planck equation. (Appendix A.I.)
h	Planck's constant.
I_{p,v_1}^{q,v_2}	Integral expression defined by Eq. (A-58).
$I_n(x)$	Integral expression defined by Eq. (B-9).
$\bar{I}(\nu_{\ell j})$	Specific intensity of $\nu_{\ell j}$-radiation integrated over all solid angles. $\bar{I}(\nu_{\ell j}) = c\rho(\nu_{\ell j})$
$\|J\|$	Jacobian of a transformation, usually defined locally.
j_e	Electron current density. [See Eq. (4).]
k	Boltzmann's constant.
m_e	Electron mass.
m_h	Mass of heavy particle.
N_0, N_{01}	Normalization constants for the distribution function.
n_e, n_h	Number densities: electrons, heavy particles.
O	Order of magnitude symbol.
$\hat{P}(v)$	Function used in approximating F_{NE}. [See Eq. (13).]
$\bar{P}_{1_{n\ell}}^{n_j}(g_{1_{n\ell}})$	Differential cross section for radiative capture. [See Eqs. (A-38) and (A-39).]
$P(x)$, $Q(x)$, $R(x)$, $S(x)$, $Q'(x)$, $R'(x)$, $Q''(x)$, $R''(x)$	Coefficients used in differential equation describing $D(x)$.
$P_m(\mu)$	Legendre polynomial of order m. [See Eq. (A-51).]
p	Total pressure of plasma.
T_e	Electron temperature, defined by Eq. (6).
T_h	Temperature of heavy particles.
T_{e_M}	Approximate electron temperature described in Section V.D.
t	Time.

v, v_m	Electron speeds.
v_D	Electron drift speed.
$\vec{v}_{mm'}$	Electron velocity defined by $[v_m, \vec{X}_{mm'}]$. (See Appendix A.I.B.)
$\vec{v}_h$	Velocity of heavy particle.
X, X_h	Parameters, associated with the degree of ionization, defined in Section III.
x	Dimensionless electron speed variable used in differential equation describing $D(x)$.
Y, Y_h	Parameters, associated with the strength of the applied field, defined in Section III.

<u>Greek Letters</u>

$\alpha_{1_{n\ell}}^{n_j}$, $\beta_{1_{n\ell}}^{n_j}$	Coefficients for spontaneous and induced radiative capture. [See Eq. (A-39).]
β, β_e	Parameters associated with heavy particle and electron temperatures. $\beta \equiv \frac{m_e}{2kT_h}$; $\beta_e \equiv \frac{m_e}{2kT_e}$
Γ_{ee}	Parameter associated with Coulombic encounters. $\Gamma_{ee} \equiv \frac{4\pi e^4}{m_e^2} \ln \bar{\Lambda}$
γ, γ_c	Magnitude of the force per unit mass felt by an electron due to an external electric field. $\gamma \equiv -\frac{eE}{m_e}$; $\gamma_c \equiv -\frac{eE}{m_e}$
$\Delta_{n_{jk}}$, $\Delta_{n_{j\ell}}$	Excitation or ionization potential associated with a particular transaction. [See Eqs. (A-2), (A-20), and (A-36).]
δ	Effective mass ratio defined by Eq. (A-54).
δ_h	Mass ratio defined by $\delta_h \equiv \frac{2m_e}{m_h}$.

δ_i, δ_n	Value of δ for a fully ionized or very weakly ionized plasma.
$\delta_{\vec{v}_3,\vec{v}}$	Delta function defined in Appendix A.I.B.
ϵ_m	Azimuthal "scattering" angle following an encounter.
ζ	Parameter defined by $\zeta \equiv \frac{8\pi v^3 \tilde{f}^0}{n_e}$.
Λ	Dimensionless parameter defined in Section III.
$\bar{\Lambda}$	Ratio of the Debye length to the average impact parameter for a 90° Coulomb deflection. $\bar{\Lambda} \equiv \lambda_D/(e^2/3kT_e)$
λ_D	Debye length. $\lambda_D \equiv \left(\frac{kT_e}{4\pi n_e e^2}\right)^{1/2}$
$\lambda(v)$	Overall mean free path for momentum transfer. [See Eq. (A-90).]
λ_E, λ_i, λ_n	Mean free paths for momentum transfer as a result of elastic encounters. [See Eqs. (A-55), (A-56), and (A-57).]
λ_{NE}	Effective mean free path for momentum transfer as a result of nonelastic encounters. [See Eq. (A-89).]
λ_{NE}^0	Effective mean free path for nonelastic processes as related to isotropic effects. [See Eq. (A-88).]
$\lambda_{IS_{jk}}$, λ_{IS_0}, λ_{IS_1}, λ_{In_j}, λ_{In_k}	Effective mean free paths for momentum transfer associated with inelastic and superelastic collisions. (See Appendix A.II.B.1.)
$\lambda_{IR_{j\ell}}$, $\lambda_{I2_{j\ell}}$, $\lambda_{R1_{\ell j}}$, $\lambda_{R_{\ell j}}$, $\lambda_{Rec_{\ell j}}$, $\lambda_{Ion_{j\ell}}$	Effective mean free paths for momentum transfer associated with ionization and three-body recombination encounters. (See Appendix A.II.B.2.)
$\lambda_{Ph_{\ell j}}$	Effective mean free path defined in Appendix A.II.B.3.

μ, μ_m	Cosines of co-latitude angles subtended by electron velocities $\vec{v}$, $\vec{v}_m$ and the electric field.
μ_i, μ_{i_n}, μ_n	Reduced mass, defined by $\mu_i \equiv \frac{m_e m_i}{m_e + m_i}$; $\mu_{i_n} \equiv \frac{m_e m_{i_n}}{m_e + m_{i_n}}$; $\mu_n \equiv \frac{m_e m_n}{m_e + m_n}$
$\nu_{n_{jk}}$	Frequency associated with excitation potential for a particular interaction. $\Delta_{n_{jk}} \equiv h\nu_{n_{jk}}$
$\nu_{\ell j}$	Radiation frequency associated with photo-ionization encounter.
ν_{ee}	Electron-electron collision frequency. $\nu_{ee} \equiv n_e \Gamma_{ee}/v^3$
ν_{En}	Electron-neutral collision frequency. $\nu_{En} \equiv v \sum_n 1/\lambda_n$
ν_{Eh}	Elastic electron-heavy particle collision frequency. $\nu_{Eh} \equiv v/\lambda_E$
ν_{NE}, ν_{NE}^0	Collision frequencies associated with non-elastic encounters. $\nu_{NE} \equiv \frac{v}{\lambda_{NE}}$; $\nu_{NE}^0 \equiv \frac{v}{\lambda_{NE}^0}$
ν_t	Collision frequency used in Frost conductivity expression. [See Eq. (38).]
$\rho(\nu_{\ell j})$	Radiant energy density at frequency $\nu_{\ell j}$.
σ, σ_F, σ_M	Electrical conductivity defined by Eqs. (5), (38), and Section V.C., respectively.
$\sigma_n(g_n,\chi)$, $\sigma_i(g_i,\chi)$	Angular distribution functions associated with elastic electron-neutral and electron-ion differential collision cross sections.
$\sigma_{n_j}^k$, $\sigma_{n_k}^j$	Angular distribution functions associated with inelastic and superelastic differential cross sections.
$\sigma_{n_j ion}^{i_{n\ell}}$, $\sigma_{3i_{n\ell}rec}^{n_j}$	Distribution functions associated with ionization and three-body recombination cross sections.

$\phi(x)$	Error function. $\phi(x) \equiv \mathrm{erf}(x)$
χ, χ_m	Angle of "deflection" of electrons with speed v, v_m .
$\vec{\chi}_{g_3, g_{CM}}$	Angle between relative velocity vectors $\vec{g}_{3i_{n\ell}}$ and $\vec{g}_{i_{n\ell}}$.
$\psi^{i_{n\ell}}_{n_j}(v_{\ell j})$	Radiation distribution function associated with the probability of photoionization encounter.
Ω	Solid angle used in describing particle direction.
ω_k, ω_j; ω_ℓ	Degeneracies associated with states of an atom (ω_j, ω_k) and its ion (ω_ℓ).

Superscripts

$(\ )'$	Quantity evaluated before an inverse collision.
$(\ )^0$	Isotropic part of a quantity.
$(\ )^1$	Anisotropic part of a quantity.

Subscripts

e	Electron.
m	Indicates association with electron of speed v_m . In text $m = 0,1,2,3,4$.
m'	Indicates association with electron of speed $v_{m'}$. $m' \neq m$
i	Ion.
n	Neutral species of type n . Neutral particles.
n_j, n_k	Neutral particles of type n in states j, k .
i_n	Ion associated with type n neutral.
$i_{n\ell}$	Ion, associated with type n neutral, in state ℓ .
h	Heavy particle. $h = i, n, n_j, n_k, i_n, i_{n\ell}$.
E	Refers to quantities associated with elastic encounters.
NE	Refers to quantities associated with nonelastic encounters.
CM	Refers to center-of-mass coordinates.
j, k, ℓ	Refers to states of a neutral particle (j,k) and its ion (ℓ).
ℓj, jk, $j\ell$	Refers to phenomena involving transition between states indicated.

Ex	Indicates association with inelastic and super-elastic encounters.
$(\ )_{Ex_{j \leftrightarrow k}}$	Refers to particular nonelastic encounter and its inverse which results in the neutral species state changing between j and k .
Ion	Indicates association with collisional ionization and three-body recombination encounters
Ph	Indicates association with photoionization and two-body recombination encounters.
$(\ )_{Ion_{j \leftrightarrow \ell}}$, $(\ )_{Ph_{j \leftrightarrow \ell}}$	Refers to particular nonelastic encounter and its inverse which results in the heavy particle changing between a neutral in state j and an ion in state ℓ .

Miscellaneous

$\equiv$	Equal by definition.
$\sim$	Approximate equality.
$\ll$	Very much less than.
$\gg$	Very much greater than.
$\leq$	Less than or equal to.
$(\vec{\ })$	Vector quantity.
$\langle\ \rangle$	Mean quantities used in Appendix A.II.B.2.
Σ	Summation.
$(\mathrm{Atom})^j_n$	Atom of type n in state j .
d^3v, d^3v_m, d^3v_h	Volume elements in velocity space.
$d\Omega$	Differential solid angle.
∇_v	Gradient in velocity space.
$\frac{\partial}{\partial v_r}$	Gradient in velocity space. Cartesian tensor notation.
$\frac{\partial}{\partial v_r} \frac{\partial}{\partial v_s}$	Tensor operator.
ΔE_{NE}	Net energy transferred to electrons via nonelastic collisions. [See Eq. (49).]
$\frac{\partial_e}{\partial t}$	Rate of change of quantity per unit volume in phase space as a result of collisions.

I. INTRODUCTION

The accurate prediction of the electron temperature, electrical conductivity, and other properties in a partially ionized plasma with differing electron and heavy-particle mean energies rests on a knowledge of the electron distribution function. Of particular interest are conditions under which departures occur from a Maxwellian form for the isotropic part of the distribution. This problem has been treated in various degrees by other authors[1,2,3,4,5]. This treatment parallels these investigations somewhat, the main differences being the plasma conditions and applications considered, the treatment of the nonelastic interactions, and the detailed results.

The purpose of the present study is to find solutions to the kinetic equation of a plasma of arbitrary degree of ionization when subjected to various electric-field strengths. To this end the electron Boltzmann equation is formulated for a spatially uniform plasma, composed of monatomic neutrals, ions and electrons, in a strong electric field. The Fokker-Planck collision operator is used for the electron-electron interactions, and the Boltzmann collision operators are used for the electron-heavy particle encounters. Collective oscillations resulting from long-range charged-particle interactions will be assumed absent. Nonelastic as well as elastic collisions are considered. The restriction to monatomic heavy particles manifests itself only in the type of nonelastic encounters allowed.

For the case of elastic collisions only, in addition to finding the distribution function, the transition of its isotropic part from the weak-ionization limit to the Maxwellian form is illustrated analytically and numerically. Calculated electrical conductivities have been compared with the experimental results of Kerrebrock and Hoffman[6] and Cool and Zukoski[7] with good agreement. Under the conditions of these experiments, no significant departures from a Maxwellian form for the isotropic part of the distribution are predicted by the numerical solutions.

At high electron temperatures the electrons can excite or ionize the neutral particles. These interactions, which are usually considered only as a possible contribution to the energy losses from a plasma via subsequent radiation, will be shown to contribute to the calculation of the isotropic and anisotropic parts of the distribution function.

The details of this work are presented in the following sections: In Section II the basic equations and their restrictions are outlined. Section III contains an order-of-magnitude analysis of these equations. Analytical solutions of these equations with and without nonelastic effects are presented in Section IV. Results of the numerical calculations are given in Section V along with some approximate forms for the isotropic part of the distribution function. Appendix A contains the details of the derivation of the basic equations outlined in Section II. A generalized Spitzer-Härm equation applicable to partially ionized gases with and without nonelastic encounters is derived in Appendix B. Appendix C contains a brief but adequate discussion on the computational procedures employed.

II. BASIC EQUATIONS

Restricting our treatment to a spatially homogeneous plasma in an electric field, the electron velocity distribution function $f_e(\vec{v},t)$, normalized to the electron number density n_e , satisfies the Boltzmann equation in the form

$$\frac{\partial f_e}{\partial t} + \vec{\gamma}\cdot\nabla_v f_e = \sum_n \left(\frac{\partial_e f_e}{\partial t}\right)_n + \sum_i \left(\frac{\partial_e f_e}{\partial t}\right)_i + \left(\frac{\partial_e f_e}{\partial t}\right)_e + \sum_n \left(\frac{\partial_e f_e}{\partial t}\right)_{Ex} + \sum_n \left(\frac{\partial_e f_e}{\partial t}\right)_{Ion} + \sum_n \left(\frac{\partial_e f_e}{\partial t}\right)_{Ph} . \tag{1}$$

In this equation $\vec{\gamma}$, the force per unit mass of the electron due to a uniform externally applied electric field, is given by $-e\,\vec{E}/m_e$. The terms on the right of (1) represent the changes in f_e due to elastic and nonelastic collisions. The first three terms refer to the elastic encounters between electrons and neutrals, ions, and other free electrons respectively. The subscripts indicate the type of particles interacting with the electrons, and the sums are over the various neutral and ionic species present. The last three collision terms represent the nonelastic interactions considered most important in collision-dominated monatomic plasmas. These respectively account for

(1) inelastic and superelastic encounters,

(2) ionization and three-body recombination encounters,

and (3) photoionization and two-body recombination encounters. The sums are over the neutral species participating in these interactions.

All heavy-particle distribution functions are assumed to be Maxwellian at the temperature T_h . The state of the heavy particles will be indicated by the subscript j, k, or ℓ added to the species subscript. These states are considered to be specified by the principal and total angular-momentum quantum numbers and degeneracies.

Following the procedure outlined in Appendix A, that is, employing a truncated expansion of f_e in Legendre polynomials and the simplifications consistent with the small electron mass, Eq. (1) with the previously indicated collision terms can be written as the following coupled equations for the isotropic part f^0 and the anisotropic part f^1 of the electron distribution function:

$$\frac{\partial f^0}{\partial t} + \frac{1}{v^2}\frac{\partial}{\partial v}\left\{ \frac{\gamma}{3} v^2 f^1 - \frac{v^4}{2\beta}\frac{\delta}{\lambda_E}\left(\frac{1}{2v}\frac{\partial f^0}{\partial v} + \beta f^0\right) - f^0 I^{0,v}_{0,0} - \frac{v}{3}\frac{\partial f^0}{\partial v}\left(I^{0,v}_{2,0} + I^{0,\infty}_{-1,v}\right)\right\} = (F_{NE} - f^0)\frac{v}{\lambda^0_{NE}} \tag{2}$$

$$\frac{\partial f^1}{\partial t} + \gamma\frac{\partial f^0}{\partial v} = -\frac{v f^1}{\lambda} + \left(\frac{\partial_e f^1}{\partial t}\right)_e . \tag{3}$$

The electron distribution function is related to f^0 and f^1 by

$$f_e(\vec{v},t) \approx f^0(v,t) + \mu f^1(v,t)$$

where μ is the cosine of the angle between $\vec{v}$ and $\vec{E}$. This formulation is valid for $f^1 << f^0$, which implies that the magnitude of the electron drift velicity v_D is much smaller than its thermal speed[2,8]. This condition will subsequently be related to the magnitude of the applied field.

Equations (2) and (3) contain the isotropic and anisotropic parts of the Boltzmann equation respectively. The first two terms of both these equations are the direct result of applying the truncated Legendre polynomial expansion to the left-hand side of (1). The second term in brackets in (2) results from the isotropic part of the elastic electron-heavy particle collisions. In this term $\beta \equiv m_e/2kT_h$, δ is a mean electron-to-heavy-particle mass ratio, and λ_E is the mean free path for elastic momentum transfer between electrons and heavy species. δ and λ_E are defined by Eqs. (A-54) and

(A-55) of the appendix. The remaining terms in the bracket correspond to the isotropic part of electron-electron interactions. The I^{q,v_2}_{p,v_1}'s are defined by Eq. (A-58). On the right-hand side of (2) are the isotropic contributions from nonelastic collisions. F_{NE}, representing a gain of electrons, and λ^0_{NE}, the effective mean free path for the nonelastic processes as related to isotropic effects, are defined by Eqs. (A-87) and (A-88). The first term on the right-hand side of (3) includes all the elastic and nonelastic collisional contributions to the anisotropic part of the Boltzmann equation. λ, the mean free path for momentum transfer, both elastically and nonelastically, between electrons and all the heavy particles, is defined by Eq. (A-90). The last term in (3) is the anisotropic part of the electron-electron interaction and is defined by Eq. (A-59).

If f^1 is known, the electron current density and the electrical conductivity of the plasma can be obtained from the expressions

$$j_e \equiv -n_e e v_D = -\frac{4\pi}{3} e \int_0^\infty v^3 f^1 dv \tag{4}$$

and

$$\sigma \equiv \frac{j_e}{E} = \frac{4\pi e^2}{3m_e\gamma}\int_0^\infty v^3 f^1 dv . \tag{5}$$

The electron temperature, given by

$$T_e \equiv \frac{m_e\overline{v^2}}{3k} = \frac{4\pi m_e}{3kn_e}\int_0^\infty v^4 f^0 dv , \tag{6}$$

will satisfy the energy equation for the electron gas. Taking the energy moment of (1), or equivalently, multiplying the isotropic equation (2) by $m_e v^2/2$ and integrating over all electron speeds yields

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_e k T_e\right) = \frac{j^2}{\sigma} - \int_0^\infty \left(\frac{m_e v^2}{2}\right)\delta\nu_{Eh}\left(\frac{1}{2\beta v}\frac{\partial f^0}{\partial v} + f^0\right) 4\pi v^2 dv + \int_0^\infty \left(\frac{m_e v^2}{2}\right)(F_{NE} - f^0)\,\nu^0_{NE}\, 4\pi v^2 dv \tag{7}$$

where $\nu_{Eh} \equiv \frac{v}{\lambda_E}$ and $\nu^0_{NE} \equiv \frac{v}{\lambda^0_{NE}}$.

III. ORDER-OF-MAGNITUDE ANALYSIS

We will limit the present analysis to the steady state and henceforth neglect the time variations in Eqs. (2), (3), and (7). Equation (2) can then be integrated directly to give

$$\frac{\gamma}{3} v^2 f^1 - \frac{v^4 \delta}{2\beta\lambda_E}\left(\frac{1}{2v}\frac{df^0}{dv} + \beta f^0\right) - f^0 I_{0,0}^{0,v} - \frac{v}{3}\frac{df^0}{dv}\left(I_{2,0}^{0,v} + I_{-1,v}^{0,\infty}\right)$$

$$= \int_0^v (F_{NE} - f^0)\frac{v^3}{\lambda_{NE}^0}\, dv \;. \tag{8}$$

Equations (3) and (7) become

$$\gamma \frac{df^0}{dv} = -\frac{vf^1}{\lambda} + \left(\frac{\partial_e f^1}{\partial t}\right)_e \tag{9}$$

$$\frac{J^2}{\sigma} = \int_0^\infty \left(\frac{m_e v^2}{2}\right) \delta\nu_{Eh}\left(\frac{1}{2\beta v}\frac{df^0}{dv} + f^0\right) 4\pi v^2 dv - \int_0^\infty \left(\frac{m_e v^2}{2}\right)(F_{NE} - f^0)$$

$$\cdot \nu_{NE}^0\, 4\pi v^2 dv \;. \tag{10}$$

Our objective will be to investigate the solutions of Eqs. (8) and (9) in various regimes differentiated by the degree of ionization and the field strength. To this end we perform an order-of-magnitude analysis of the coupled equations for f^0 and f^1 which will serve to assess the range of validity and illustrate the confluence of these restricted solutions. In particular, when considering elastic collisions only, it is possible to show the transition of f^0 between the forms applicable to the weak and fully ionized limits.

With respect to order of magnitude, we take f^0 as a Maxwellian distribution function at the electron temperature; that is,

$$f^0 \sim n_e\left(\frac{\beta_e}{\pi}\right)^{3/2} e^{-\beta_e v^2} \equiv \tilde{f}^0 \tag{11}$$

where $\beta_e = m_e/2kT_e$.

Since in (8) all of the electron-electron collision terms are of the same order of magnitude, we replace them by a typical term, namely, the first one, which for $f^0 = \tilde{f}^0$ can be shown to be

$$I_{0,0}^{0,v} = \nu_{ee} v^3 \Lambda \tag{12}$$

where

$$\Lambda = \phi(x) - x\phi'(x),\; x = \sqrt{\beta_e}\, v,\; \phi(x) = \text{erf}\,(x)$$

and where $\nu_{ee} \equiv n_e \Gamma_{ee}/v^3$ is the electron-electron collision frequency. Under the $f^0 = \tilde{f}^0$ approximation, we can replace F_{NE} in the nonelastic collision terms by $\tilde{f}^0\hat{P}$, where $\hat{P}$ is a measure of the inequality between the actual number densities of the interacting species and their equilibrium values at the electron temperature. $\hat{P}$ is given by

$$\hat{P}(v) = \Bigg\{ \sum_{n,k,j}\left[\frac{n_{n_k}}{n_{n_j}}\left(\frac{n_{n_j}}{n_{n_k}}\right)_{Eq}\frac{1}{\lambda_{I_{nj}}} + \frac{n_{n_j}}{n_{n_k}}\left(\frac{n_{n_k}}{n_{n_j}}\right)_{Eq}\frac{1}{\lambda_{I_{nk}}}\right]$$

$$+ \sum_{n,\ell,j}\left[\frac{n_{i_{n\ell}} n_e}{n_{n_j}}\left(\frac{n_{n_j}}{n_{i_{n\ell}} n_e}\right)_{Eq}\frac{1}{\lambda_{Ion_{j\ell}}} + \frac{n_{n_j}}{n_e n_{i_{n\ell}}}\left(\frac{n_e n_{i_{n\ell}}}{n_{n_j}}\right)_{Eq}\frac{1}{\lambda_{Rec_{\ell j}}}\right]$$

$$+ \sum_{n,\ell,j}\frac{n_{n_j}}{n_e n_{i_{n\ell}}}\left(\frac{n_e n_{i_{n\ell}}}{n_{n_j}}\right)_{Eq}\frac{1}{\lambda_{Ph_{\ell j}}}\Bigg\}\, \lambda_{NE}^0 \;. \tag{13}$$

The new symbols that appear in this equation have been defined in Appendix A; those subscripted by Eq are the equilibrium values mentioned above. The summations are over the neutral species n , their excited states j, k, and the states ℓ of their related ions. Under quasi-equilibrium conditions at T_e , $\hat{P}$ is equal to unity. With these simplifications we can represent (8) by

$$\frac{2}{3}\gamma v^2 f^1 \sim \tilde{f}^0 \delta v^3 \nu_{Eh}\left(1-\frac{\beta_e}{\beta}\right) + \tilde{f}^0 2\nu_{ee} v^3 \Lambda + 2\int_0^v (\hat{P}-1)\frac{\tilde{f}^0 v^3}{\lambda^0_{NE}}\, dv \;. \tag{14}$$

We treat (9) similarly, replacing $\left(\frac{\partial e f^1}{\partial t}\right)_e$ by the first term in Eq. (A-59) and using Eq. (A-90). The result is

$$f^1 \sim \frac{2\beta_e v\gamma \tilde{f}^0}{\nu_{Eh}+\nu_{NE}-8\pi\nu_{ee}\frac{v^3\tilde{f}^0}{n_e}} \tag{15}$$

where $\nu_{NE} = v/\lambda_{NE}$.

The combination of equations (14) and (15) yields

$$\frac{4/3\,\beta_e\,\gamma^2}{\nu_{Eh}+\nu_{NE}-\zeta\nu_{ee}} \sim \delta\left(1-\frac{\beta_e}{\beta}\right)\nu_{Eh} + 2\Lambda\nu_{ee} + \frac{2}{\tilde{f}^0 v^3}\int_0^v (\hat{P}-1)\frac{\tilde{f}^0 v^3}{\lambda^0_{NE}}\, dv \tag{16}$$

where $\zeta \equiv \frac{8\pi v^3 \tilde{f}^0}{n_e}$. The left-hand side of this equation, which is proportional to the square of the electric field strength, represents the forcing mechanism for the nonequipartition state. The terms on the right-hand side represent respectively the effects of the electron-heavy, electron-electron and nonelastic collisions in the isotropic equation. The various collision terms in the anisotropic equation are shown explicitly in (15) and in the denominator of the left-hand side of (16).

Now, using the definitions

$$X_h \equiv \frac{\nu_{ee}}{\nu_{Eh}} \qquad \text{and} \qquad Y_h \equiv \frac{4}{3}\frac{\gamma^2\beta_e}{\nu^2_{Eh}} \;,$$

Eq. (16) can be written as

$$\frac{Y_h}{1+\frac{\nu_{NE}}{\nu_{Eh}}-\zeta X_h} \sim \delta\left(1-\frac{\beta_e}{\beta}\right) + 2\Lambda X_h + 2\frac{1}{\tilde{f}^0 v^3}\frac{1}{\nu_{Eh}}\int_0^v (\hat{P}-1)\tilde{f}^0 v^2 \nu^0_{NE}\, dv. \tag{17}$$

It is also convenient to introduce the alternate parameters X and Y . X, defined as the ratio of the electron-electron collision frequency to the total electron-neutral collision frequency ν_{En} , is related to the degree of ionization. The relation between X_h and X is

$$X_h = \frac{X}{1+X} \;. \tag{18}$$

Y , defined as the square of the ratio of the average energy gained by an electron between collisions in an electric field to the electron thermal energy, is approximately equal to the square of the ratio of the electron drift speed to the mean thermal speed. Y_h , a measure of the strength of the electric field, is related by Y by the expression

$$Y_h \sim (1+X_h)^2\, Y \;. \tag{19}$$

Since $0 \leq X_h \leq 1$, we can conclude that $Y_h \sim Y$.

Performing a similar order-of-magnitude analysis on the energy equation, utilizing here a mean-free-path expression for the conductivity and constant collision frequencies throughout, yields

$$Y \sim \left[\delta\left(1-\frac{\beta_e}{\beta}\right) + \frac{\nu^0_{NE}}{\nu_{Eh}}\left(1-\frac{4}{3}\frac{\beta_e}{n_e m_e}\int_0^\infty \left(\frac{m_e v^2}{2}\right)\tilde{f}^0\hat{P}\, 4\pi v^2 dv\right)\right]\left(\frac{1+X}{1+2X}\right) \;. \tag{20}$$

For a fully ionized gas our model gives $\nu^0_{NE} = 0$, and Eq. (10) can be written as

$$Y \approx 3\bar{\delta}_i(1-\beta_e/\beta) \tag{21}$$

which agrees with (20). $\bar{\delta}_i \equiv \sum_i n_i\delta_i/n_e$ is the value of δ for the fully ionized gas.

Although by definition X, X_h, and Y_h are velocity-dependent, henceforth they will be considered as being mean values evaluated at the electron thermal speed.

In the following section we will use Eqs. (17)-(20) in presenting solutions to Eqs. (8) and (9) for various values of the parameters X and Y . Before doing so, we investigate the limits on X and Y. From its definition X can vary between 0 and ∞ corresponding to a very weakly ionized or a fully ionized plasma. Y, however, which has a lower limit of 0 for the case when there is no applied field, has a maximum dictated by the condition $f^1 << f^0$. This condition causes Y_{max} to satisfy the inequality $(Y_{max})^{1/2} << 1$.

For the fully ionized plasma a static instability resulting from the energy equation places additional restrictions on the maximum value of Y [4,9]. Using Eq. (21), on writing Y in terms of T_e and E, it is readily found that the maximum applied field for a stable fully ionized uniform plasma can be represented by $Y_{max} \approx \delta_1$. This condition corresponds to $T_e = 3/2\ T_h$.

For a very weakly ionized plasma, undergoing elastic collisions only, (20) can be used to show that when the electron-neutral cross sections do not decrease with temperature more rapidly than $1/(T_e\sqrt{1-T_h/T_e})$ (this is the case for the neutrals we consider here), the strength of the applied field is not limited, and the electron temperature increases monotonically with E. Y for this case, however, will have a maximum given by $\bar{\delta}_n$, its value as T_e approaches infinity. $\bar{\delta}_n$ is the value of δ for a plasma that is dominated by neutral collisions. Similar results with δ replacing $\bar{\delta}_n$ apply for any plasma that is not Coulomb-collision dominated[10]. Inclusion of the nonelastic collisions will usually lower the electron temperature for the same value of the field strength, as a result of the additional mechanism for the transfer of energy from the electrons to the neutral particles and possible radiation losses. This will have the effect of increasing the maximum possible value of Y for a partially ionized gas. As the magnitude of this increase is not easily estimated, we will take as the upper limit on Y for a partially ionized gas, based on energy considerations, the value previously obtained when only elastic encounters were considered.

Based on the above analysis, the limiting maximum on Y for any particular problem will be the most stringent of the two choices available. That is, for any degree of ionization, either $(Y_{max})^{1/2} << 1$ or $Y_{max} \lesssim \delta$, the value of δ depending on the degree of ionization. For many monatomic gases δ is less than 10^{-4}, and it appears that the two criteria are complementary.

We note that Y_h can be related to the discharge parameter E/p and to E_c, the critical electric field for runaway in a fully ionized gas[11], by the following expressions:

$$Y_h = (E/p)^2 \frac{2}{3} \frac{e^2}{mkT_e} \frac{p^2}{\nu_{Eh}^2} \tag{22}$$

and

$$Y_h \sim \gamma^2/\gamma_c^2 , \tag{23}$$

where p is the total pressure of the mixture and $\gamma_c \equiv eE_c/m$. From (23) and the limits on Y_{max} we see that we consider only electric fields for which $\gamma << \gamma_c$ in this analysis.

IV. SOLUTIONS

We first consider the solution of (8) and (9) for the case where only elastic collisions are important. Later in this section the effects of the nonelastic terms are discussed.

A. Elastic Collisions Only

When the nonelastic terms are neglected, (8) and (9) become

$$\frac{\gamma}{3} v^2 f^1 = \frac{\delta}{\lambda_E} \frac{v^4}{2\beta}\left(\frac{1}{2v}\frac{df^0}{dv} + \beta f^0\right) + f^0 I_{0,0}^{0,v} + \frac{v}{3}\frac{df^0}{dv}\left(I_{2,0}^{0,v} + I_{-1,v}^{0,\infty}\right) \tag{24}$$

and

$$\gamma \frac{df^0}{dv} = -\frac{vf^1}{\lambda_E} + \left(\frac{\partial_e f^1}{\partial t}\right)_e . \tag{25}$$

The solution of these equations for various values of the parameters X and Y can best be described with the aid of Fig. 1, where we have taken Y_{max} as δ. This figure qualitatively illustrates the regions of applicability of these solutions.

Expressions (17) and (20) with the nonelastic contributions deleted become

$$\frac{Y_h}{1-\zeta X_h} \sim \delta\left(1-\frac{\beta_e}{\beta}\right) + 2\Lambda X_h \tag{26}$$

$$Y \sim \delta\left(1-\frac{\beta_e}{\beta}\right)\left(\frac{1+X}{1+2X}\right) . \tag{27}$$

1. Equipartition of Energy

Values of Y very much less than δ correspond to weak electric fields and thus to equal electron and heavy-particle temperatures as can be seen from expression (27). Accordingly, only the collisional terms remain in (26) or in the isotropic equation (24) for this case. The following identity, which is true for $f^0 = \tilde{f}^0$ at any T_e, is easily verified[5]:

$$\frac{4\beta_e v^2}{3}\left(I_{2,0}^{0,v} + I_{-1,0}^{0,\infty}\right) = 2I_{0,0}^{0,v} . \tag{28}$$

Thus (24), for weak electric fields, is satisfied by a Maxwellian distribution function at the gas temperature. That is, for $Y \ll \delta$ and for any X we find

$$f^0 = n_e\left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} \equiv \tilde{f}_T^0 . \tag{29}$$

For weak fields, the weak ionization limit occurs when $X \ll 1$. The fact that ζ in (26) has a maximum of about 7.1 indicates that we can neglect the electron-electron collision terms in (25) for this range of X. Thus, in this weak ionization limit we find

$$f^1 = -\frac{\lambda_E \gamma}{v}\frac{df^0}{dv} \tag{30}$$

which becomes

$$f^1 = 2\gamma\lambda_E\beta\tilde{f}_T^0 \tag{31}$$

when Eq. (29) is utilized. The electrical conductivity can be evaluated for this case by combining (31) with (5).

Spitzer and Härm's[12] results for fully ionized plasmas apply whenever $X \gg 1$. This can be illustrated as follows: If the denominator of the left-hand side of (26) is written in terms of X, via (18), it becomes apparent that we can neglect the electron-neutral collisions in this case. Then (26), with $f^0 = \tilde{f}_T^0$ can be written as

$$2\beta v\gamma\tilde{f}_T^0 = vf^1 \sum_i \frac{1}{\lambda_i} - \left(\frac{\partial_e f^1}{\partial t}\right)_e . \tag{32}$$

If now we represent f^1 as a multiple of f^0, that is, $f^1 = D(v)f^0$, and set $\sum_i 1/\lambda_i = \nu_{ee}/v$, (32) can be reduced to

$$\frac{d^2D}{dx^2} + P(x)\frac{dD}{dx} + Q(x)D = R(x) + S(x) \tag{33}$$

where $x = \sqrt{\beta}v$. This equation is identical to the one solved numerically by Spitzer and Härm. The terms $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are identical to the P, Q, R and S of Spitzer and Härm for the case when their mean ionic charge factor equals unity. [In Appendix B we derive the Spitzer-Härm equation for the more general case of a partially ionized gas. This development leads to Eqs. (36) and (37) below.] The conductivity can be found in terms of $D(x)$ by means of (5). Spitzer and Härm tabulate $D(x)$ and give the following expression for the conductivity:

$$\sigma = \frac{2}{(\pi\beta)^{3/2}} \frac{0.582\, m_e}{e^2 \ell n\, \bar{\Lambda}} . \tag{34}$$

$\bar{\Lambda}$ is the ratio of the Debye length to the average impact parameter for a 90° Coulomb deflection.

In the region between the weak ionization and the fully ionized limits (25) becomes

$$2\beta v \gamma \tilde{f}_T^0 = \frac{v f^1}{\lambda_E} - \left(\frac{\partial_e f^1}{\partial t}\right)_e . \tag{35}$$

Again setting f^1 equal to $f^0 D$, Eq. (35) can be reduced to an equation which is identical in form to (33), with Q and R replaced respectively by

$$Q'(x) \equiv Q(x) - \frac{2}{\Lambda} \frac{\nu_{En}(x)}{\nu_{ee}(x)} \tag{36}$$

and

$$R'(x) \equiv R(x) - \frac{16x^4}{3\sqrt{\pi}\Lambda}\left(1 - \frac{6}{5}x^2\right) \int_0^\infty \frac{\nu_{En}(x)}{\nu_{ee}(x)} e^{-x^2} D\, dx, \tag{37}$$

where

$$\nu_{En}(x) \equiv \frac{x}{\sqrt{\beta}} \sum_n \frac{1}{\lambda_n} .$$

This resulting equation can be solved by the Chapman-Enskog method, that is, by expanding $D(x)$ in a series of Laguerre polynomials and utilizing their orthogonality properties, and the electrical conductivity can be accurately determined. This has been done by Shkarofsky[5]. Schweitzer and Mitchner[13] solved the equivalent problem using the Boltzmann collision operator for the electron-electron interactions. In addition, the latter authors verify the accuracy of an empirical mixture rule by Frost[14] for the calculation of the electrical conductivity in this region. The Frost conductivity expression for

this case is

$$\sigma_F = -\frac{4\pi e^2}{3m_e} \int_0^\infty \frac{v^3}{\nu_t} \frac{df^0}{dv}\, dv \tag{38}$$

where

$$\nu_t \equiv \nu_{En} + \frac{0.952}{v^2} n_e \Gamma_{ee} \sqrt{\beta} .$$

We have examined this equation for several experimental cross sections and confirmed Schweitzer and Mitchner's observations. These results will be presented later in Section V.

2. Nonequipartition

In the nonequipartition case, when $Y \ll \delta$, the preferential energy addition to the electrons by the electric field is sufficiently great that the mean electron energy exceeds the mean heavy-particle energy. The energy equation, when nonelastic effects are neglected, can be written as

$$\frac{j^2}{\sigma} = \int_0^\infty \left(\frac{m_e v^2}{2}\right) \delta \nu_{Eh}\left(\frac{1}{2\beta v}\frac{df^0}{dv} + f^0\right) 4\pi v^2 dv . \tag{39}$$

To evaluate this equation or Eq. (6) for the electron temperature, we need to find f^0.

The weak ionization limit for strong fields occurs for $X \ll \delta$. We see from (26) that for this case we can neglect electron-electron collisions in both the isotropic and anisotropic equations. Then Eqs. (24) and (25) can be combined into a first-order linear differential equation for f^0 which can be solved to yield

$$f^0 = N_0 \exp\left\{ -\int_0^v \frac{2\beta v dv}{\left(1 + \frac{4}{3}\frac{\beta\gamma^2\lambda_E^2}{\delta v^2}\right)} \right\} . \tag{40}$$

N_0 is a constant of the integration and can be evaluated by normalizing f_e to n_e. Since the electron-electron and the total electron-ion collision frequencies are of the same order, the electron-ion collision terms can also be neglected when evaluating δ and λ_E. In (40) the term $\frac{4}{3}\frac{\beta\gamma^2\lambda_E^2}{\delta v^2}$ can be approximated by $\frac{\beta}{\beta_e}\frac{Y_h}{\delta}$. Thus, for a weak field the distribution is Maxwellian at the gas temperature, while for a strong field it can differ significantly from a Maxwellian distribution. For hard-sphere molecules in a strong electric field, $\beta >> \beta_e$, (4) reduces to the well-known Druyvesteyn distribution[15]. For any arbitrary cross section the anisotropic part f^1 of the distribution can be obtained from Eq. (30). The electrical conductivity, (5), then becomes

$$\sigma = -\frac{4\pi e^2}{3m_e}\int_0^\infty \frac{v^3}{v_{Eh}}\frac{df^0}{dv}\,dv\ . \qquad (41)$$

In the less restrictive case, when $X << 1$, we can neglect the electron-electron collisions only in the anisotropic equation; f^1 is again given by (30). Equations (30) and (24) can be combined into a first-order integro-differential equation which can be integrated to yield

$$f^0 = N_0\exp\left\{-\int_0^v \frac{\left(1+\frac{2\lambda_E}{\delta v^4}I_{0,0}^{0,v}\right)2\beta v dv}{1+\frac{4\beta\lambda_E}{3\delta v^2}\left(I_{2,0}^{0,v}+I_{-1,v}^{0,\infty}+\gamma^2\lambda_E\right)}\right\}\ . \qquad (42)$$

N_0 is again the normalization constant for f_e. This equation can be solved by an iterative technique. We have done this numerically and will discuss these results later. The electrical conductivity for this case can be found by the use of the solution to (42) in Eq. (41).

An examination of (26) reveals that whenever $X >> \delta$, corresponding to a partially ionized gas, and $Y \leq \delta$, the only significant terms remaining in the isotropic equation are the electron-electron collision terms. That is, (24) reduces to

$$f^0 I_{0,0}^{0,v} + \frac{v}{3}\frac{df^0}{dv}\left(I_{2,0}^{0,v}+I_{-1,v}^{0,\infty}\right) = 0\ . \qquad (43)$$

The solution to this equation is, as may be verified by substitution and use of (28),

$$f^0 = \tilde{f}^0 \equiv n_e\left(\frac{\beta_e}{\pi}\right)^{3/2} e^{-\beta_e v^2}\ . \qquad (44)$$

That is, the isotropic part of the distribution function satisfying (43) is Maxwellian although at an elevated temperature. The electron temperature is determined from (39) which can be written now as

$$\frac{J^2}{\sigma} = 2\pi m_e\left(1-\frac{\beta_e}{\beta}\right)\int_0^\infty \frac{\delta}{\lambda_E} v^5\,\tilde{f}^0\,dv\ . \qquad (45)$$

The equation for f^1 for this case becomes

$$2\beta_e v\gamma\tilde{f}^0 = \frac{vf^1}{\lambda_E} - \left(\frac{\partial_e f^1}{\partial t}\right)_e\ . \qquad (46)$$

Equation (46) is identical to (35) with β replaced by β_e.* Since the temperature in the anisotropic equation appears only in the Maxwellian f^0, results found for the weak-field case can be validly extended into the strong-field region simply by replacing T_h by T_e. Thus, between $X >> \delta$ and the fully ionized limit, the results of the Chapman-Enskog expansion can be extended to strong fields in this manner. In accord with the results of Schweitzer and Mitchner, we would then expect the Frost conductivity expression with β_e replacing β to give accurate approximate results for strong electric fields. This concept has been tested numerically and will be discussed later.

The fully ionized gas, $X >> 1$, is a special case of the situation just described. Spitzer and Härm's solution for $D(x)$ with x redefined as $\sqrt{\beta_e}v$, is then valid in the strong-field region. Accordingly, the electrical conductivity is given by (34) evaluated at the electron temperature. The electron temperature is as before obtained from the energy equation (45), which for this case is

*An equation identical to (33) with Q and R replaced by Q' and R' from (36) and (37) and with x defined as $\sqrt{\beta_e}v$ could be derived from (46).

$$T_e = T_h + \left(\frac{3\pi}{16(0.582)}\right) \frac{m_e J^2}{e^2 n_e^2 3k \, \bar{\delta}_1} \, . \tag{47}$$

B. Nonelastic Effects

In this subsection we present formal solutions to (8) and (9). These solutions are then related to the previous elastic results.

1. Preliminary Comments

Equation (17) allows us to draw the following general conclusions:

a. If $\nu_{NE}^0 \ll \delta\nu_{Eh}$ or $\nu_{NE}^0 \ll \nu_{ee}$ at the energy where the nonelastic collisions may be significant, the nonelastic effects can be neglected and the elastic results remain valid.

b. If $\nu_{NE} \ll \nu_{Eh}$ at all energies, the nonelastic collisions can be neglected in the momentum equation. For cases in which this holds, f^0 would be determined as outlined below and f^1 would be found from (25).

It is difficult to assess nonelastic effects in other than a gross sense unless a specific model of nonelastic behavior is chosen. We confine our remarks to three classes of nonelastic behavior characterized by whether for each nonelastic interaction* the rate of upward induced transitions (A) exceeds, (B) is less than, or (C) is equal to the frequency of inverse downward transitions.

For class (A) we expect the net energy transferred to the electrons by nonelastic interactions ΔE_{NE} to be negative corresponding to electron kinetic energy being transferred to potential energy of the bound electronic states of the heavy particles. Class (B) behavior would result in ΔE_{NE} being positive. Class (C) would correspond to no net energy transfer by nonelastic collisions.

In a real plasma, depending upon the constituents, their energy, and other physical parameters, any combination of the

*Described in the sense of Eqs. (A-2,3,20,21, and 36.)

classes can exist. Class (A) behavior when caused by radiation escape forms an interesting problem in high-temperature plasmas.

Before proceeding with the general analysis we examine qualitatively the following nonelastic terms which appear in the isotropic and energy equations:

$$\frac{1}{2} c(v) \equiv \int_0^v (F_{NE} - f^0) \frac{v^3}{\lambda_{NE}^0} dv \text{ , and} \tag{48}$$

$$\Delta E_{NE} \equiv 4\pi \int_0^\infty \left(\frac{m_e v^2}{2}\right)(F_{NE} - f^0) \frac{v^3}{\lambda_{NE}^0} dv \, . \tag{49}$$

The electron continuity equation for the steady state results in

$$c(\infty) = 0 \, . \tag{50}$$

An investigation of the form of F_{NE} confirms that the integrands of both $c(v)$ and ΔE_{NE} vanish for all v when the plasma is in equilibrium or $f^0 = \tilde{f}^0$ and Boltzmann statistics at T_e apply. In these cases detailed balancing occurs and the nonelastic interactions do not influence the isotropic part of the distribution directly. However, as will be discussed later, the nonelastic interactions may still contribute to the evaluation of f^1 and thus the current density, the electrical conductivity, and indirectly to f^0 through T_e via the energy equation.

The fact that (50) always holds implies that whenever ΔE_{NE} is negative, $(F_{NE} - f^0)$ must (on the average) be positive for low values of v and negative for very large values of v. Hence, we expect $c(v) \geq 0$ when it is not identically zero for all v. This last statement, which is true if $(F_{NE} - f^0) = 0$ has but one root for $v > 0$, may not be correct for complicated interactions in which many different threshold energies exist; however, on the average it should be valid. Similarly, for class (B) behavior, when ΔE_{NE} is positive we expect $c(v) \leq 0$. By definition class (C) behavior corresponds to $c(v) = 0$. Henceforth, we will consider the designations $c(v) \geq 0$, $c(v) \leq 0$ and $c(v) = 0$ as synonymous with the classes (A), (B) and (C) respectively.

2. Weak-Field Case

When the plasma is exposed to weak electric fields such that $Y << \delta$, we can neglect the left-hand side of Eq. (17). This corresponds to writing (8) as

$$\frac{df^0}{dv} + \frac{b}{a} f^0 + \frac{c}{a} = 0 \tag{51}$$

where

$$a(v) = \frac{1}{2\beta v}\left[\frac{\delta}{\lambda_E} v^4 + \frac{4}{3}\beta v^2 \left(I_{2,0}^{0,v} + I_{-1,v}^{0,\infty}\right)\right] ,$$

$$b(v) = \frac{\delta}{\lambda_E} v^4 + 2 I_{0,0}^{0,v} \text{ , and}$$

$c(v)$ is given by (48).

Equation (51) can be integrated to yield

$$f^0 = \exp^{-\int_0^v \frac{b}{a} dv} \left[N_0 - \int_0^v \frac{c}{a} \exp^{\int_0^v \frac{b}{a} dv} dv\right] \tag{52}$$

which reduces to (29) when $c(v) = 0$. N_0 is the normalization constant. For $Y << \delta$, we previously found that $T_e = T_h$ when nonelastic effects were neglected. Their presence here when $c(v) \geq 0$, however, has the effect of lowering T_e below T_h. Physically in this situation electrons lose energy to the heavy particles during the nonelastic encounters and gain energy from the heavy particles during elastic encounters.

Analytically, $c(v) \geq 0$ should manifest itself in a change in f^0 away from $\tilde{f}_T^0$. In particular, we would expect the tail of the distribution to be depressed relative to the Maxwellian distribution at the gas temperature[16]. To illustrate this effect, we let the solution to (51) be given by

$$f^0 = \tilde{f}_T^0 A(v) \tag{53}$$

where $\tilde{f}_T^0$ is the solution when $c(v) = 0$. Combining Eqs. (53)

and (51) results in the following equation for $A(v)$:

$$\frac{dA}{dv} = -\frac{c}{a\tilde{f}_T^0} . \tag{54}$$

Integrating this equation we obtain

$$A(v) = A(0) - \int_0^v \frac{c}{a\tilde{f}_T^0} dv \tag{55}$$

where $A(0)$ is determined from the normalization of f^0. The integrand in Eq. (55) is positive (on the average), increasing and then decreasing with v. Thus $A(v)$ decreases nearly monotonically with v from $A(0)$ and, as expected, the tail of f^0 will be depressed relative to $\tilde{f}_T^0$.

For this case, if $c(v) \leq 0$ the tail of f^0 would be elevated relative to $\tilde{f}_T^0$; the electrons would gain energy via nonelastic encounters and lose energy elastically, and the electron temperature would be greater than the gas temperature.

Now, if in addition to considering only weak fields here, we further restrict ourselves to $X << 1$ (the weak-ionization limit), Eq. (9) can be written as

$$f^1 = -\frac{\lambda\gamma}{v}\frac{df^0}{dv} . \tag{56}$$

The electrical conductivity can then be obtained from (5).

In a fully ionized gas, based on our model which neglects multiple ionizations and excitation interactions between electrons and ions, we have $c(v) = 0$ and therefore the elastic results [(32) to (34)] apply directly.

Equation (9) for f^1 applies between the weak-ionization and fully ionized limits. When $c(v) = 0$, corresponding to an optically thick plasma, this equation can be written as

$$\frac{d^2D}{dx^2} + P(x)\frac{dD}{dx} + Q''(x)D = R''(x) + S(x) \tag{57}$$

where

$x = \sqrt{\beta} v$,

$$Q''(x) = Q'(x) - \frac{2}{\Lambda} \frac{\nu_{NE}(x)}{\nu_{ee}(x)} \text{ , and} \tag{58}$$

$$R''(x) = R'(x) - \frac{16x^4}{3\sqrt{\pi}\,\Lambda} (1 - \frac{6}{5} x^2) \int_0^\infty \frac{\nu_{NE}(x)}{\nu_{ee}(x)} e^{-x^2} D \, dx \ . \tag{59}$$

Equation (57) can be solved by the Chapman-Enskog expansion technique. [A method of deriving Eqs. (57), (58) and (59) is described in Appendix B.] When $c(v) \neq 0$ the equations become much more complicated and will not be discussed here. In line with the mixture rule of Frost (38) presented with the elastic results, one might expect a similar mixture rule with ν_t replaced by $\nu_t + \nu_{NE}$ to give reasonable results for the conductivity between the $X \ll 1$ and the fully ionized limits.

3. Strong-Field Case

As can be seen from the energy equation and the sign of ΔE_{NE}, the inclusion of the nonelastic collisions in the analysis changes T_e relative to its elastic value. Once f^0 is determined, (6) can be used to find T_e.

As was found to be true for the weak-field region, in each case to be discussed below the effect on the isotropic part of the distribution function on nonelastic interactions when $c(v) \geq 0$ is to depress the tail of f^0 relative to its $c(v) = 0$ counterpart. The opposite effects will occur when $c(v) \leq 0$.

For any degree of ionization and strong fields (8) and (9) can be combined into a linear first-order integro-differential equation for f^0 which when integrated results in an equation identical in form to (52) with $a(v)$ and $b(v)$ given in Table I.

In the weak ionization limit all Coulombic interactions are neglected and f^0 is given by equation (52) with $a(v)$ and $b(v)$ from Table I. Here when $bf^0 \gg c$ or $\nu_{En} \gg \nu^0_{NE}$, (52) reduces to the form of the elastic solution (40). The presence of λ rather than λ_E in $a(v)$, however, causes

TABLE I. Coefficients for Eq. (52)

	$a(v)$	$b(v)$
Weak ionization limit	$\frac{1}{2\beta v}[\frac{5}{\lambda_E} v^4 + \frac{4}{3}\beta v^2 \gamma^2 \lambda]$	$\frac{5}{\lambda_E} v^4$
Weakly ionized	$\frac{1}{2\beta v}[\frac{5v^4}{\lambda_E} + \frac{4}{3}\beta v^2 (I^{0,v}_{2,0} + I^{0,\infty}_{-1,v} + \gamma^2\lambda)]$	$\frac{5v^4}{\lambda_E} + 2I^{0,v}_{0,0}$
Partially ionized	$\frac{2}{3} v (I^{0,v}_{2,0} + I^{0,\infty}_{-1,v})$	$2I^{0,v}_{0,0}$

the solution to differ from (40). Since λ is less than λ_E, even if $c(v) = 0$ the existence of nonelastic collisions whenever the electric-field term in the isotropic equation is significant in the evaluation of f^0 would still tend to depress the tail of the distribution relative to the elastic solution. The electrical conductivity (5) for this case is evaluated using f^1 as obtained from (56) with the electron-ion interactions neglected in λ.

As was the case for elastic collisions only, when $X \ll 1$, corresponding to a weakly ionized plasma, we can neglect the electron-electron collisions in (9), and f^1 is again given by (56). This equation when combined with (8) and the result integrated yields (52) with a and b from Table I for f^0. When $c(v) = 0$, this result reduces in form to its elastic counterpart, Eq. (42). A comparison of ν_{NE} and ν_{NE}^0 with ν_{Eh} will indicate the relative importance of the nonelastic collision terms. The electrical conductivity for this case is determined by combining Eqs. (56) and (5).

In a partially ionized plasma, for which $X \gg \delta$ and $Y \leq \delta$, the electron-heavy particle collision terms and the field term can be neglected in the isotropic equation. This equation can then be integrated to yield (52) with $a(v)$ and $b(v)$ from Table I. When $c(v) = 0$, this result reduces to (44). From the differential equation (8) it is apparent that the nonelastic terms can be neglected when $bf^0 \gg c$ or $\nu_{ee} \gg \nu_{NE}^0$.

In an optically thick, partially ionized plasma [$c(v)=0$] the nonelastic terms do not contribute to the calculation of the isotropic part of the distribution function; however, they can affect the evaluation of the transport properties as a result of the momentum transferred between electrons and the heavy particles during the nonelastic collisions.

This effect can be illustrated by evaluating the effective mean free path for momentum transfer as a result of nonelastic

collisions under these conditions, where use of Boltzmann statistics and the Saha equation, evaluated at the electron temperature, are also valid. To simplify the development we will work in the region on Fig. 1 between $X \gg \delta$ and $X \ll 1$. In this region Eqs. (56) and (44) applied to each nonelastic interaction yield the following relation between the isotropic and anisotropic parts of the electron distribution function:

$$\frac{f_m^1}{f^1} = \frac{\lambda(v_m) f_m^0}{\lambda(v) f^0} . \tag{60}$$

The subscript m, corresponding to similar subscripts appearing in Eq. (A-87) and other subsequent equations in Appendix A, will take on integer values between 0 and 4 depending on the energy of the free electrons participating in the various interactions. For inelastic and superelastic collisions, use of Eq. (60) and the Boltzmann relations in Eq. (A-69) yields

$$\frac{1}{\lambda_{IS_{jk}}(v)} = n_{n_j} \int \sigma_{n_j}^k \left(1 - \frac{\lambda(v_0)}{\lambda(v)} \cos \chi_{0_{CM}}\right) d\Omega_{0_{CM}} + n_{n_k} \int \sigma_{n_k}^j \left(1 - \frac{\lambda(v_1)}{\lambda(v)} \cos \chi_{1_{CM}}\right) d\Omega_{1_{CM}} .$$

where

$$v_0 = \sqrt{v^2 - \frac{2}{m_e} \Delta_{n_{jk}}} \quad \text{and} \quad v_1 = \sqrt{v^2 + \frac{2}{m_e} \Delta_{n_{jk}}} .$$

The effective mean free path for momentum transfer resulting from ionization and recombination interactions, Eq. (A-79), becomes on use of the Saha equation and (60):

$$\frac{1}{\lambda_{IR_{j\ell}}(v)} = n_{n_j} \int_0^{v_{4max}} \iint \left(1 - \frac{\lambda(v_2)}{\lambda(v)} \cos \chi_{2_{CM}} - \frac{\lambda(v_4)}{\lambda(v)} \cos \chi_{4_{CM}}\right) \cdot \sigma_{n_j ion}^{i_{n\ell}} \, d\Omega_{2_{CM}} d\Omega_{42_{CM}} v_4^2 dv_4 + n_{i_{n\ell}} \int_0^{v_{3max}} \iint \left(1 - \frac{\lambda(v_1)}{\lambda(v)} \cos \chi_{1_{CM}} + \frac{\lambda(v_3)}{\lambda(v)} \cos \chi_{v_3 v}\right) v_3^3 f_3^0 \sigma_{3 i_{n\ell} rec}^{n_j} d\Omega_{1_{CM}} d\Omega_{3_{CM}} dv_3$$

with v_2 and v_1 given by

$$v_2 = \sqrt{v^2 - v_4^{\,2} - \frac{2}{m_e}\Delta_{n_{j\ell}}}$$

and

$$v_1 = \sqrt{v^2 + v_3^{\,2} + \frac{2}{m_e}\Delta_{n_{j\ell}}} \; .$$

Except for the $\lambda(v_m)/\lambda(v)$ ratios, the above equations for $\lambda_{IS_{jk}}$ and $\lambda_{IR_{j\ell}}$ have the familiar structure of momentum transfer mean free paths for elastic collisions. If no net momentum were transferred via nonelastic encounters, we would expect $1/\lambda_{NE} = 0$ for all v . [If $1/\lambda_{NE} \neq 0$ the net momentum transfer as a result of nonelastic interactions could vanish for particular combinations of λ_{NE} and f^1 which result in

$$\int_0^\infty v^3 \nu_{NE} f^1 dv = 0 \; .$$

Since $f^1(v)$ and $\nu_{NE}(v)$ are independent, this would generally not be the case.] The effective mean free path for momentum transfer as a result of photoionization and its inverse is greater than zero.

From their above forms, both $1/\lambda_{IS_{jk}}$ and $1/\lambda_{IR_{j\ell}}$ would not generally equal zero even if we assume $1/\lambda_{NE} = 0$ in their evaluation. This fact coupled with $\lambda_{Ph} > 0$ leads to a contradiction of the assumption that $1/\lambda_{NE} = 0$. Thus $1/\lambda_{NE} \neq 0$ and the nonelastic collisions contribute to the calculation of f^1 [via their contribution to λ in (56)] even though they do not directly influence the isotropic properties.

It is interesting to note that for some $\lambda(v)$ behavior it would be possible here to obtain negative nonelastic contributions to the total collision frequency. This could be interpreted as a gain of momentum by the electrons in the range dv about v via the nonelastic encounters and result in a probable enhancement of the conductivity.

Although illustrated for a limited range of X , the same conclusions are expected to hold for all degrees of ionization in an optically thick plasma for which $f^0 = \tilde{f}^0$. The electron temperature is also affected in the above optically thick case through the Joule heating term in the energy equation (45). To determine f^1 in this case (9) can be cast into the form of Eqs. (57), (58), and (59) with x defined as $\sqrt{\beta_e}v$. Thus, as in the case of elastic collisions only, the results found in the weak-field region can be extended into the strong-field region by simply replacing T_h by T_e . The mixture rule proposed earlier becomes similarly applicable for these nonequipartition cases.

Based on our model, a fully ionized gas is described here by the strong-field results for elastic encounters as in the weak-field case.

With the use of an order-of-magnitude estimate (52), for a weakly ionized plasma, can be shown to go over to the equations which are valid on the extremities of the region of its applicability.

V. NUMERICAL RESULTS

Equation (42) has been solved numerically by an iteration technique, and the results have been used to find the electrical conductivity, current, density, and electron temperature for various ionized gases. The plasmas considered were pure argon, potassium-seeded argon, and potassium-seeded helium. The seed fractions indicated on the figures include potassium ions as well as neutrals.

The form of f^0 for various field strengths and degrees of ionization has been investigated numerically when nonelastic terms were neglected and will be discussed below. In addition, other calculations were performed to test the validity of an approximation to (42). In the region of Fig. 1 where $X \ll 1$, a comparison has been made between the use of (42) and a Maxwellian distribution for the calculation of the electrical conductivity and the electron temperature. We have also compared the calculated conductivity-current density characteristics with the experiments of Cool and Zukoski and Kerrebrock and Hoffman.

In presenting numerical results we have restricted ourselves to the consideration of only elastic encounters, and we have assumed that the number densities were either known or determinable via the Saha equation evaluated at the electron temperature. While use of the Saha equation is correct only for equilibrium conditions, it gives results which are indicative of what one might expect in a realistic situation. To include the nonelastic terms quantitatively would require a simultaneous solution of the rate equations for the state populations of the individual species with Eq. (1). Preliminary results of the effect of a non-Maxwellian distribution function on the rate equations are presented in Reference 17.

A. <u>Evolution of f^0</u>

In Fig. 1 we see that the region in which (42) is valid

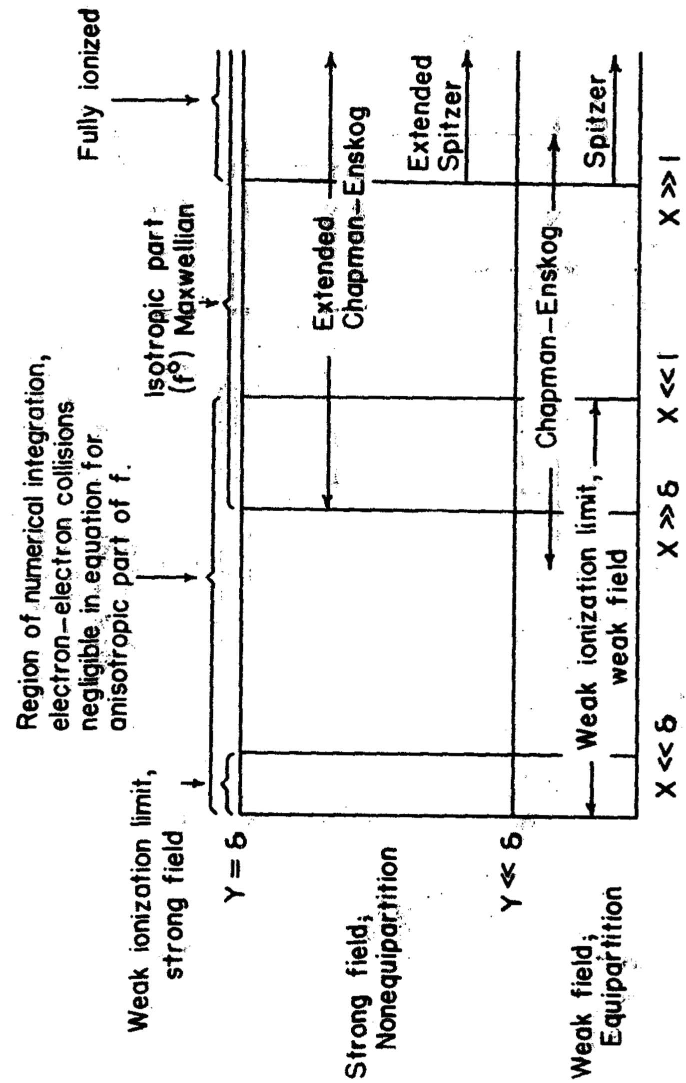


Figure 1. Qualitative Description of Nonequipartition Plasmas.

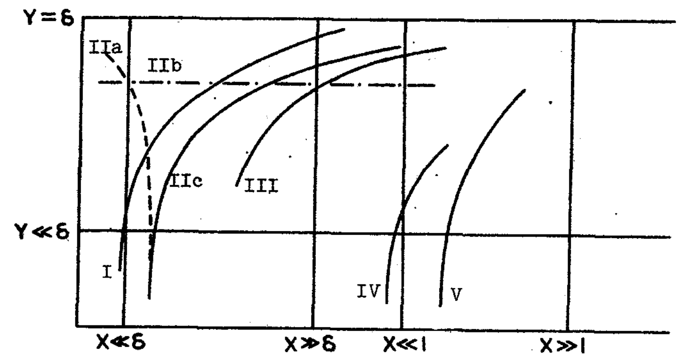
overlaps the regions in which other forms for f^0 [Eqs. (29), (40), and (44)] are also valid. It is possible, by an order-of-magnitude analysis, which makes use of (12) and (28), to cast (42) into a form which reveals the variation of f^0 with X and Y. In particular, we can demonstrate the evolution of f^0 from its form in the weak-ionization limit to the Maxwellian form applicable when $X \gg \delta$.

This transition of f^0 between the aforementioned limits has been studied by numerical integration of (42) in the region where $X \ll 1$. Typical results of this computation appear in Figs. 3, 4, 5, and 6. In such calculations the energy equation was used to find the electron temperature in the limit whenever the distribution function was Maxwellian.

Figure 2 shows typical variations of X and Y as the electric field is increased at a constant degree of ionization (----), as the degree of ionization is increased for a fixed electric field strength (— • —), or what is more realistic, as the electric field is increased and the degree of ionization is adjusted to correspond to the Saha equation at the mean electron energy as the electrons become more energetic (———). The character of f^0 along these trajectories is determined according to the region within which the trajectory lies.

By fixing the degree of ionization at a relatively low level, we illustrate in Fig. 3 the transition of the distribution function from the weak-field (gas-temperature Maxwellian) case to the strong-field (Lorentzian) case. This result is for the He-K system corresponding to curve IIa of Fig. 2.

Figure 4, corresponding to curve IIb of Fig. 2, shows how the distribution evolves from the strong field, weak ionization limit to the strong field, partially ionized case. (Here the electron number densities were chosen so the conditions for curve III in Fig. 4 would be similar to the 10 v/cm curve of Fig. 5.) The crossing of the curves on Fig. 4 at about 1 ev indicates that, for roughly the same electron temperature, the Lorentzian distribution function is depressed



I: Pure Argon, $T_h = 3000°K$, $p = 0.1$ atm.
II: He-K, $n_K/n_{He} = 0.001$, $T_h = 1250°K$, $p = 1$ atm.
III: A-K, $n_K/n_A = 0.001$, $T_h = 1250°K$, $p = 1$ atm.
IV: He-K, $n_K/n_{He} = 0.0032$, $T_h = 2000°K$, $p = 1$ atm.
V: A-K, $n_K/n_A = 0.004$, $T_h = 2000°K$, $p = 1$ atm.

— — — e-e collisions neglected, number densities fixed at equilibrium values corresponding to T_h.

Figure 2. Approximate Variation of X and Y for the Cases Considered.

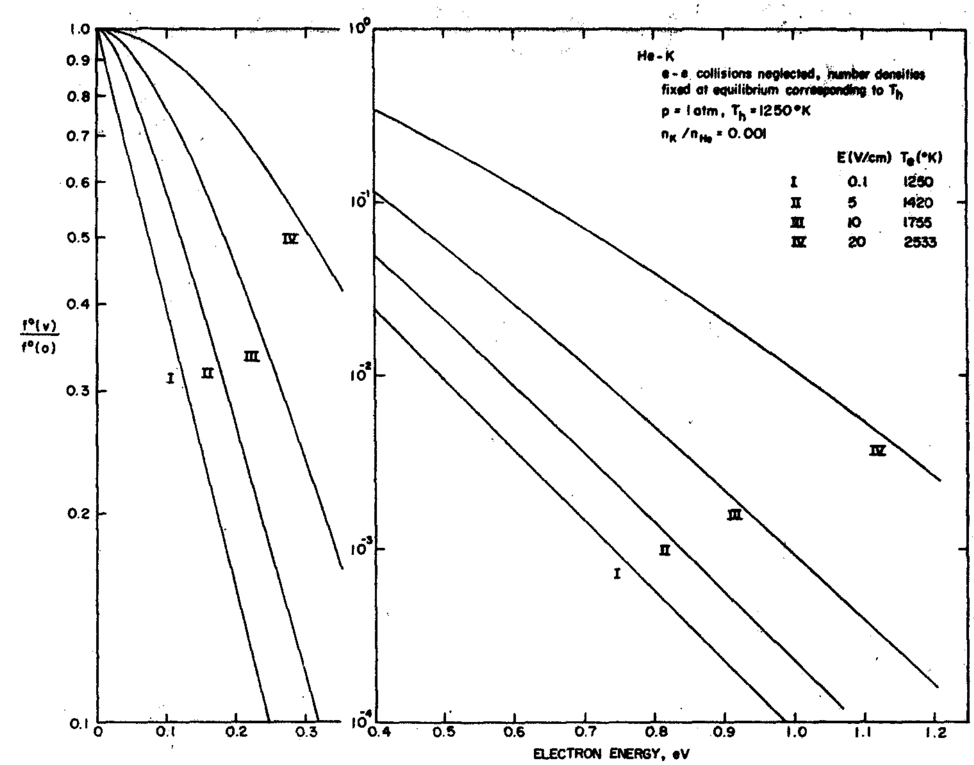


Figure 3. Evolution of f^0 with Increasing Electric Field in the Weak Ionization Limit.

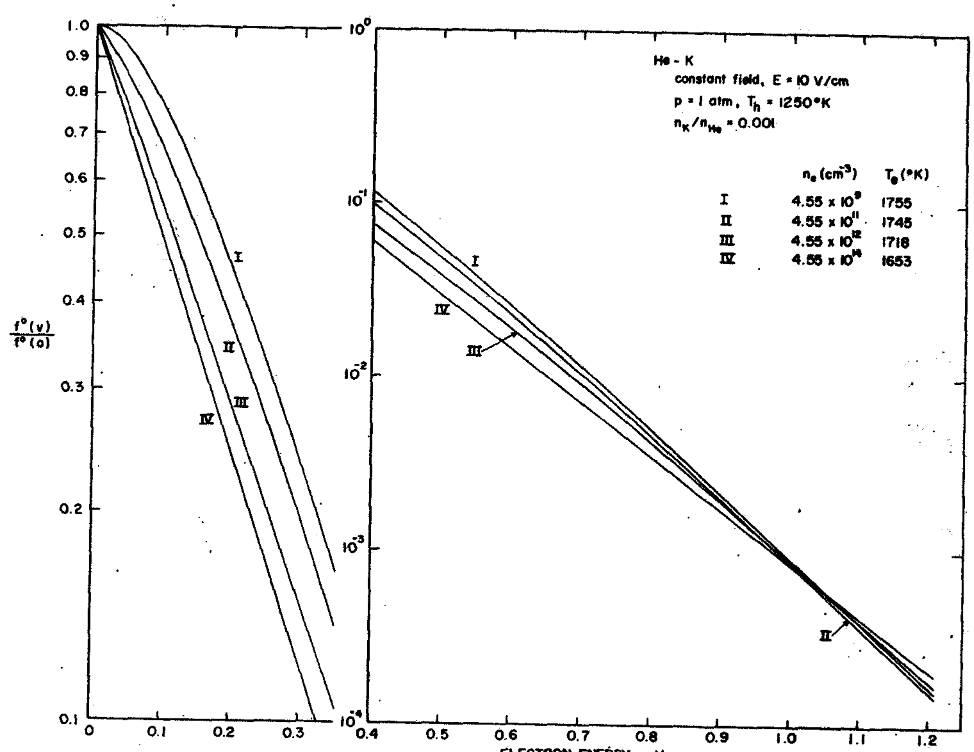


Figure 4. Evolution of f^0 for a Fixed Field Strength from its Form I in the Weak Ionization Limit to the Maxwellian Form IV at the Electron Temperature.

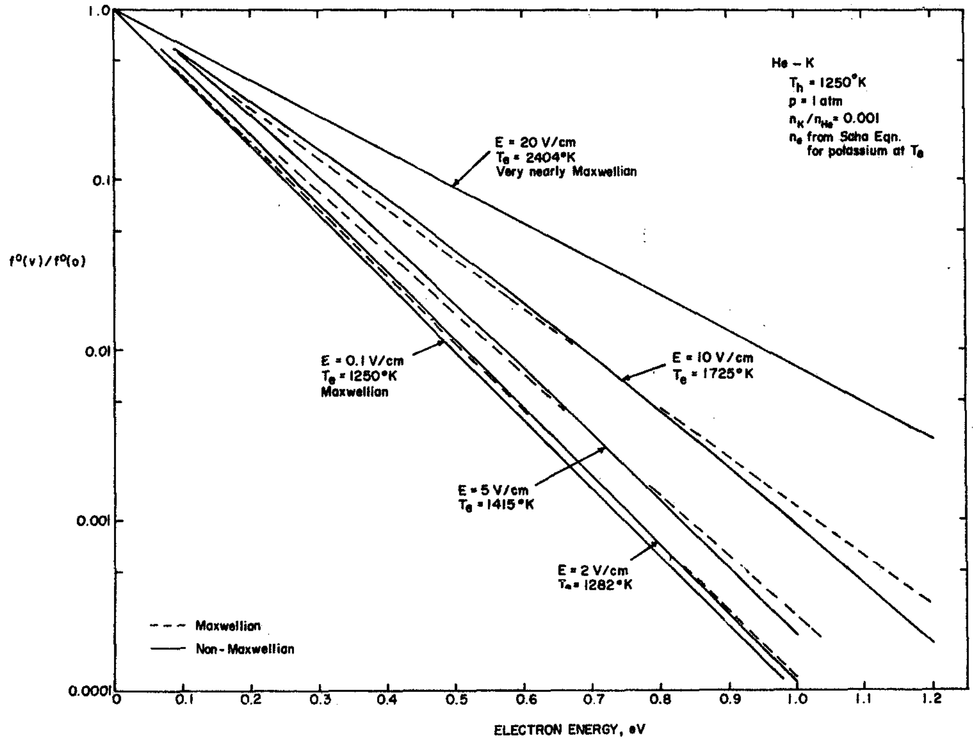


Figure 5. Evolution of the Electron Distribution Function from a Gas-temperature Maxwellian to an Electron-temperature Maxwellian. Helium-Potassium.

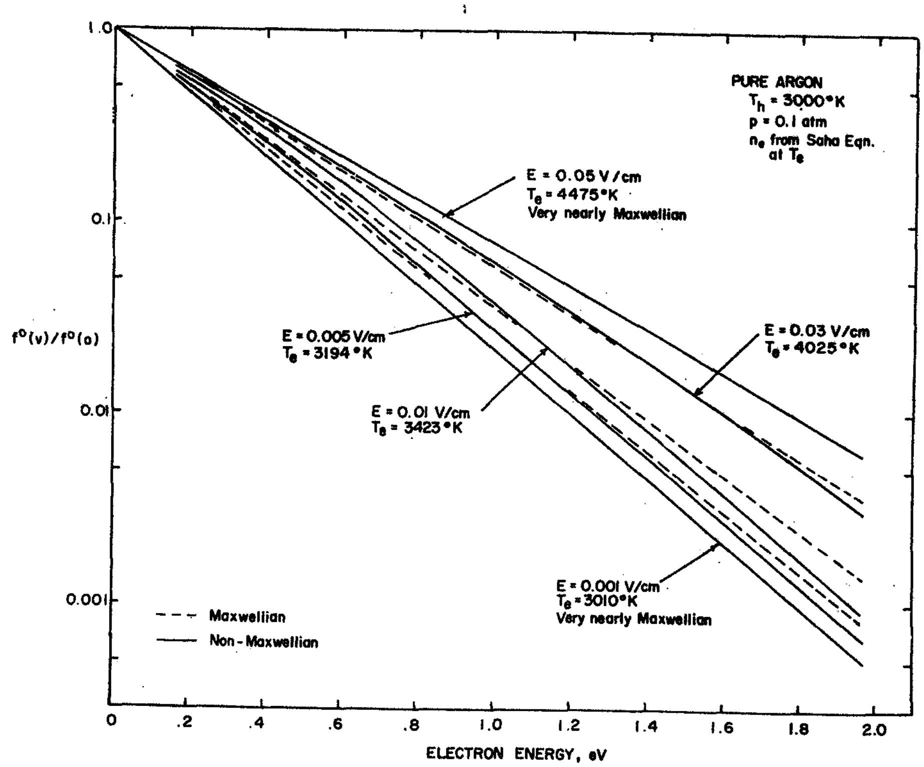


Figure 6. Evolution of the Electron Distribution Function from a Gas-temperature Maxwellian to an Electron-temperature Maxwellian. Argon.

in the high-energy tail relative to a Maxwellian distribution function.

Figure 5 illustrates the evolution of the distribution function from a gas-temperature Maxwellian to an electron-temperature Maxwellian, which occurs as the degree of ionization increases [curve IIc, Fig. 2]. Figure 6, corresponding to curve I of Fig. 2, illustrates this transition for pure argon. The A-K system corresponding to curve III of Fig. 2 also undergoes a similar transition; however, it exhibits less pronounced non-Maxwellian characteristics as might be expected as a result of its location on Fig. 2. The differences in the cross sections and the masses of helium and argon cause the displacement between curves IIc and III on this figure. Another consequence of these differences is that a stronger applied field is necessary for He compared to A to achieve significant electron heating.

B. Approximate Forms for f^0

As a result of the complexity of (42), Ginzburg and Gurevich[4], considering $\tilde{f}^0$ as the zeroth approximation to f^0, proposed the following first-order solution as a convenient representation for f^0:

$$f^{01} = N_{01} \exp\left\{ -\int_0^v \frac{\left(1 + \frac{2\lambda_E}{\delta v^4} n_e \Gamma_{ee} \Lambda\right) 2\beta v dv}{\left(1 + \frac{\beta}{\beta_e} \frac{2\lambda_E}{\delta v^4} n_e \Gamma_{ee}\Lambda + \frac{4}{3}\frac{\beta}{\delta}\frac{\lambda_E^2 \gamma^2}{v^2}\right)} \right\}. \qquad (61)$$

If $\tilde{f}^0$ is substituted into the right-hand side of (42) Eq. (61) follows. Equation (61) has all the characteristics of f^0 and agrees exactly with (42) in each of the weak-ionization, the weak-field, and the Coulomb-dominated limits. Only the electron-electron interaction terms in (42) were directly affected by this approximation. Since these terms, when they are significant, tend to Maxwellianize the distribution, the use of $\tilde{f}^0$ in evaluating them might be expected

to yield reasonable results in the regions between the aforementioned limits. The approximation f^{01} has been compared to f^0 numerically, in the region where f^0 differed significantly from a Maxwellian form, for several gases. Typical results are presented in Fig. 7. For each case the electron number density is based on the Saha equation at the temperature calculated on the basis of f^0 or f^{01}. Observe that the approximate form yields a somewhat different electron temperature than that obtained from the exact calculation and that the respective number densities on the basis of the Saha equation are significantly different.

Since the solutions presented in Section IV-B are so complex, it would be useful, even at the sacrifice of some accuracy, to develop approximate forms for the distribution function analogous to (61) which account for nonelastic encounters.

All of the solutions, when nonelastic effects are included, can be written in the form

$$f^0 = g^0(v)B(v) \qquad (62)$$

where

$$B(v) \equiv 1 - \int_0^v \frac{c}{ag^0} dv \qquad (63)$$

and

$$g^0 \equiv N_0 \exp\left\{ - \int_0^v \frac{b}{a} dv \right\}. \qquad (64)$$

In particular for the case of a weakly ionized plasma, if we assume that the zeroth approximation for the distribution function is given by $\tilde{f}^0$, then we find from (52) and Table I that

$$b \to b' \equiv \frac{\delta v^4}{\lambda_E} + 2n_e\Gamma_{ee}\Lambda , \qquad (65)$$

$$a \to a' \equiv \frac{1}{2\beta v}\left[\frac{\delta v^4}{\lambda_E} + \frac{\beta}{\beta_e} 2n_e\Gamma_{ee}\Lambda + \frac{4}{3}\beta v^2\gamma^2\lambda\right] \qquad (66)$$

and

$$c \to c' \equiv 2\int_0^v (\hat{P}-1)\, \tilde{f}^0 \frac{v^3}{\lambda^0_{NE}} dv . \qquad (67)$$

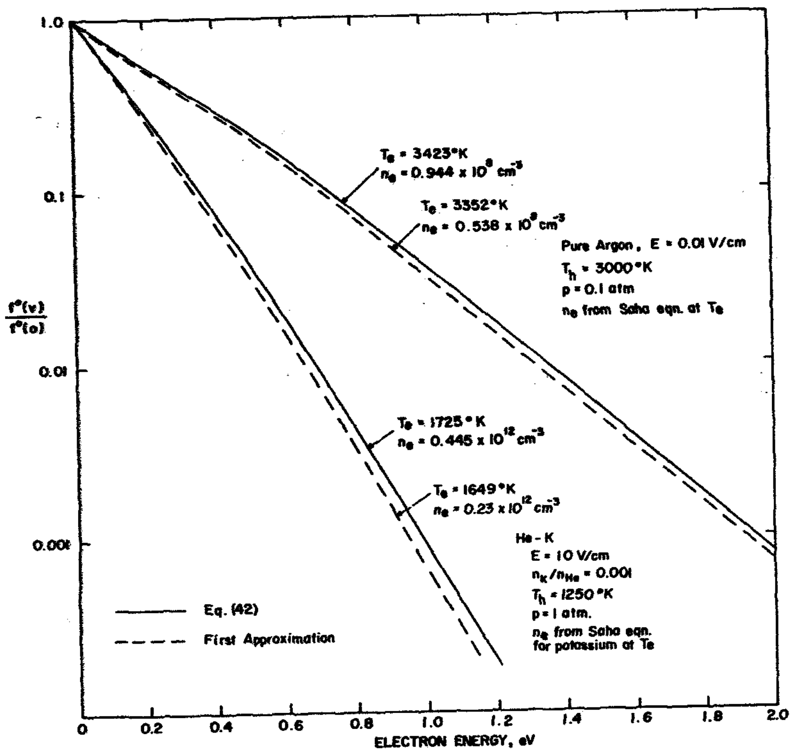


Figure 7. Comparison of f^0 with the First Approximation.

Usually $\lambda_E(v) < \lambda_{NE}(v)$ which implies $\frac{\lambda_E}{2} < \lambda < \lambda_E$. Thus $\lambda = O(\lambda_E)$ and we can approximate λ by λ_E in a' and then with (61) obtain

$$\frac{g^0}{N_0} \cong \frac{f^{01}}{N_{01}} . \tag{68}$$

Combining these results, a first approximation to (62) can be written as

$$f^{01}_{NE} = \left(\frac{f^{01}}{N_{01}}\right) \left[N_{NE} - \int_0^v \frac{c'}{a'_E\left(\frac{f^{01}}{N01}\right)} dv\right]. \tag{69}$$

In this equation (f^{01}/N_{01}) is the normalized elastic approximation given by (61), subscript NE is to emphasize that this is a nonelastic approximation, N_{NE} is the normalization constant for f^{01}_{NE}, and a'_E is a' with λ replaced by λ_E.

C. Conductivity

Figure 8 shows a comparison of conductivities σ and σ_M calculated with (41) using respectively the nonequilibrium distribution function (42) and the Maxwellian distribution $\tilde{f}^0$. The same electron temperature and the same electron density are used for both σ and σ_M. Thus the ordinate is essentially an electron-mobility ratio. The abscissa is a measure of the applied field. The dotted curves on Fig. 8 (and Fig. 9) correspond to a constant degree of ionization. As the field strength is increased, a constant degree of ionization represents the maximum departure of the distribution function from a Maxwellian. Thus these dotted curves illustrate a measure of the maximum departure of the approximate conductivity from the actual one calculated on the basis of the nonequilibrium distribution function.

The solid curves on this figure have the degree of ionization calculated according to the Saha equation. They indicate that for the cases considered the simpler σ_M provides satisfactory values for the conductivity provided that the

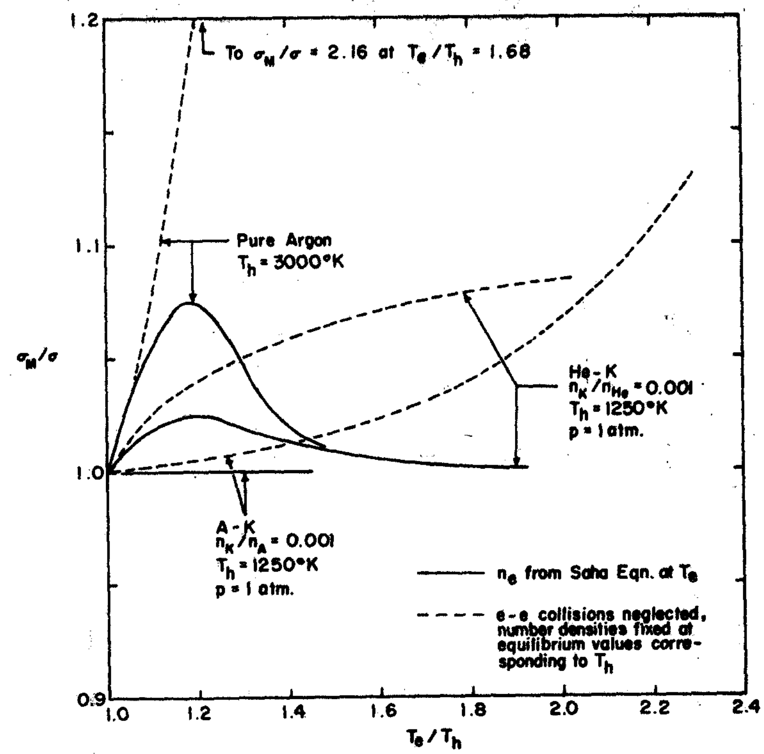


Figure 8. Comparison of the Electrical Conductivity with an Approximation Calculated on the Basis of a Maxwellian f^0.

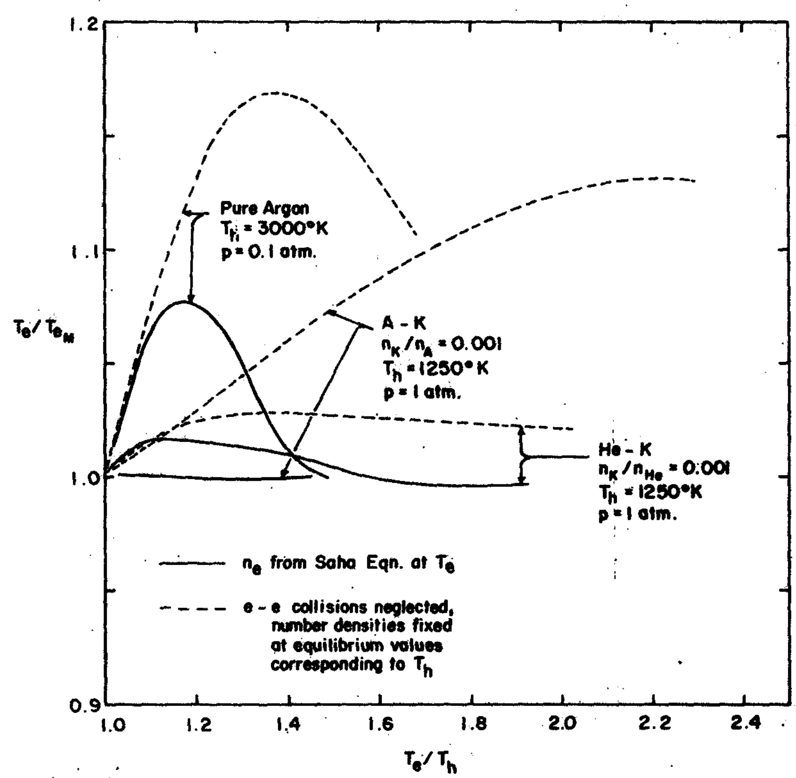


Figure 9. Comparison of the Electron Temperature with an Approximation Calculated on the Basis of a Maxwellian f^0.

electron temperature is known. As expected at weak and strong field strengths, when f^0 is Maxwellian, $\sigma_M/\sigma \rightarrow 1$. The greatest deviation between σ and σ_M occurs between these limits when f^0 differs significantly from $\tilde{f}^0$. For the A-K system it was found that $\sigma \approx \sigma_M$ as expected from the remarks in Section V-A.

D. Electron Temperature

The electron temperature T_e found by the use of the nonequilibrium distribution function (42) in (6) was compared to an approximate electron temperature T_{e_M} found by the use of an energy balance with f^0 assumed to be Maxwellian at the electron temperature. Results of this comparison are illustrated in Fig. 9. As was the case in Fig. 8, the dotted curves here represent the maximum departure of T_{e_M} from T_e. It should be noted that the relatively small differences between T_e and T_{e_M} shown here for the solid curves correspond to large differences in electron density.

E. Comparison with Experiment

Under the conditions of the experiments of Refs. 6 and 7, $X \gg \delta$, and hence if only elastic encounters are considered, we would expect the distribution function to be $\tilde{f}^0$. Numerical calculations were performed which verified that the distribution function was in fact Maxwellian during all the conditions of these experiments. The evolution of f^0 for some typical experimental conditions is shown qualitatively in Fig. 2 by curves IV and V. Figure 10 illustrates a typical comparison between experiment, the conductivity based on Frost's theory [Eq. (38)] (shown by Ref. 13 to be preferable to the mean-free-path conductivity used by Ref. 7), and the conductivity as calculated by Cool and Zukoski. Here we see no significant disagreement between the theories and good agreement between theories and experiment for the higher current densities. Similar results have been recently reported

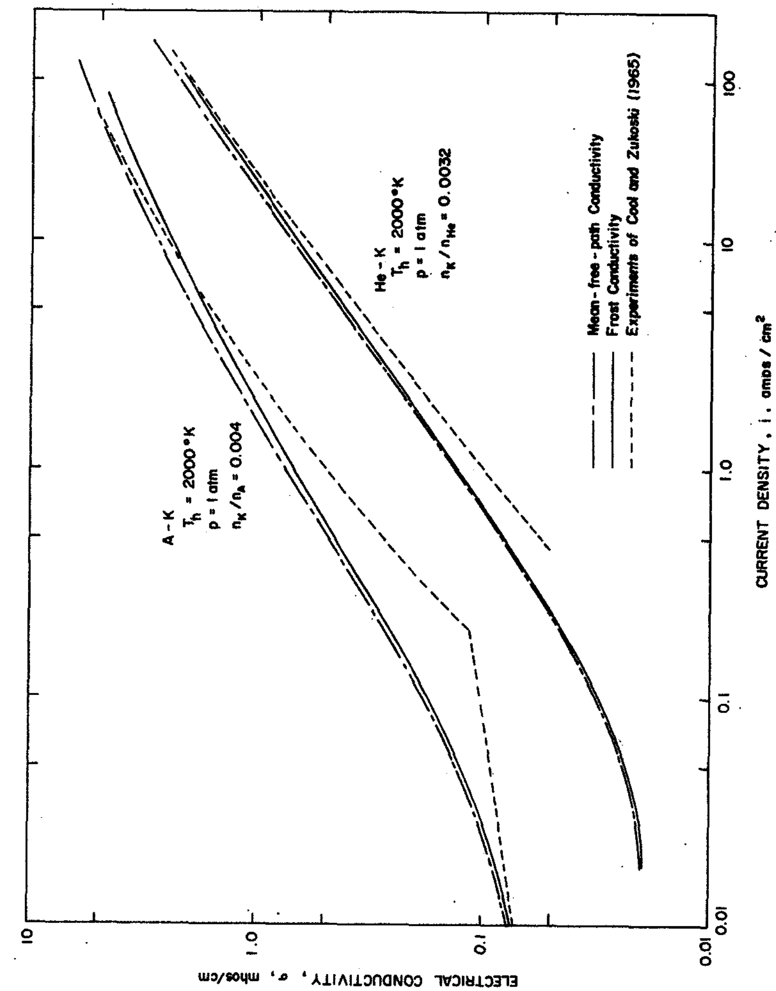


Figure 10. Electrical Conductivity-Current Density Characteristics, Comparison of Theory and Experiment.

by Lyman[18]. Inclusion of radiation loss terms in the Maxwellian energy balance has resulted in much better agreement in the region where the elastic theory does not agree well with the experimental results[6,7]. Although a similar behavior can be qualitatively deduced from the nonelastic aspects of our analysis, detailed calculations are beyond the scope of our present objectives.

APPENDIX A

DEVELOPMENT OF KINETIC EQUATIONS (2) and (3)

In this section we derive the kinetic equations which are fundamental to the previous analysis.

I. Collision Integrals

In Eq. (1) we use the familiar Boltzmann collision operator to represent the elastic electron-neutral and electron-ion encounters and the Fokker-Planck collision operator for the electron-electron interactions. These terms are discussed in several references[2,4,5] and will merely be described briefly here.

Electron-neutral:

A typical electron-neutral collision term is described by the Boltzmann binary collision integral as

$$\left(\frac{\partial_e f_e}{\partial t}\right)_n = \iint (f_n' f_e' - f_n f_e) g_n \sigma_n(g_n, \chi)\, d\Omega\, d^3 v_n \ .$$

The primed quantities denote dependency on "after-collision" velocities. The quantity $\sigma_n(g_n,\chi)d\Omega$ denotes the differential cross section for the elastic encounter between electrons and neutrals of relative speed $g_n \equiv |\vec{v} - \vec{v}_n|$ such that the relative velocity (or the electron) will be deflected through the angle χ, into the differential solid angle $d\Omega$, in the center-of-mass frame.

Electron-ion:

Here as in Ref. 2 we neglect collective plasma oscillations and consider only those Coulombic interactions which are characterized best by random two-body encounters. These encounters can be described by either the Fokker-Planck equation or Boltzmann binary collision operator[5]. For convenience we choose the latter and write

$$\left(\frac{\partial_e f_e}{\partial t}\right)_i = \iint (f_i' f_e' - f_i f_e) g_i \sigma_i(g_i, \chi)\, d\Omega\, d^3 v_i \ .$$

The notation here is consistent with that used in the previous electron-neutral collision integral. $\sigma_1(g_1,\chi)$ is the Rutherford cross section for a Coulombic interaction.

In evaluating this collision integral, we will make use of the Debye shielding length to "cut off" the divergent integrals associated with the Coulombic potential[19,20]. This cut-off procedure is valid whenever the number of particles in a Debye sphere is very much greater than unity; that is, when $\ell n\bar{\Lambda}$ is large.

Electron-electron:

Here, as in the previous Coulombic interaction, we also have a choice of collision operators. We follow the standard procedure and use the Fokker-Planck equation to account for the electron-electron Coulombic encounters. Written in terms of the now familiar Rosenbluth potentials, this collision operator in Cartesian tensor notation is

$$\left(\frac{\partial_e f_e}{\partial t}\right)_e = -\frac{\partial}{\partial v_r}\left[f_e(\vec{v})\,\frac{\partial H(\vec{v})}{\partial v_r}\right] + \frac{1}{2}\,\frac{\partial}{\partial v_r}\,\frac{\partial}{\partial v_s}\left[f_e(\vec{v})\,\frac{\partial}{\partial v_r}\,\frac{\partial}{\partial v_s}\,G(\vec{v})\right],$$

where

$$H(\vec{v}) = 2\Gamma_{ee}\int \frac{f_e(\vec{v}_f)}{g}\,d^3v_f$$

$$G(\vec{v}) = \Gamma_{ee}\int f_e(\vec{v}_f)\,g\,d^3v_f\ .$$

In the potential functions H and G,

$\vec{v}_f$ is the velocity of the field electron,

$g \equiv |\vec{v} - \vec{v}_f|$ is the relative speed between the test and field electrons, and

$\Gamma_{ee} \equiv 4\pi \frac{e^4}{m_e^2}\,\ell n\bar{\Lambda}$ where e = electron charge, and $\bar{\Lambda}$ is the ratio of the Debye length to the average impact parameter for a 90° Coulomb deflection.

Here in writing Γ_{ee} in terms of $\ell n\bar{\Lambda}$ use was made of the Debye cut-off to evaluate the Coulombic scattering cross section and the resulting logarithm was treated as a constant evaluated at the mean value of the electron velocity[21]. Dreicer[2] presents some useful relationships between H, G, and f_e.

In treating the nonelastic collisions below, we will follow the development presented by Fowler[22] but allow for thermal motion of the neutrals and work in velocity rather than energy variables.

A. Inelastic and Superelastic Collisions

We investigate the conservation of electrons in the set $\vec{v}, d^3v$ as a result of inelastic and superelastic encounters resulting in neutrals undergoing excitation and de-excitation reactions respectively. A typical interaction can be represented by

$$e + (\mathrm{Atom})_n^j \rightleftarrows e + (\mathrm{Atom})_n^k \qquad \text{(A-1)}$$

where the subscript n stands for the type of neutral atom and the superscripts j and k $(k > j)$ correspond to the state of the neutral particle.

The energy equation for this type of encounter can be distinguished by whether the reaction is caused by or results in an electron in the set. When an excitation $(j \rightarrow k)$ is caused by an electron with velocity $\vec{v}$ the energy equation is

$$m_e v^2 + m_n v_{n_j}^2 = m_e v_0^2 + m_n v_{n_k}^2 + 2\Delta_{n_{jk}}\ . \qquad \text{(A-2)}$$

Here $\vec{v}_0$ is the electron velocity after the collision, $\vec{v}_{n_j}$ and $\vec{v}_{n_k}$ are the velocities of the neutral before and after the encounter, respectively (the subscripts j and k serve the dual role of distinguishing between the neutral before and after collision and representing its state), and $\Delta_{n_{jk}} \equiv h\nu_{n_{jk}}$ is the excitation energy for the interaction. When an excitation results in an electron being added to the set, the energy equation is

$$m_e v_j^2 + m_n v_{n_j}^2 = m_e v^2 + m_n v_{n_k}^2 + 2\Delta_{n_{jk}} \qquad \text{(A-3)}$$

where $\vec{v}_1$ is the velocity of the electron before the inelastic collision. The inverse (superelastic) encounters which also contribute to the number of electrons in the set are described energetically by (A-2) and (A-3).

The number of electrons lost to the set $\vec{v}, d^3v$ in unit time per unit volume as a result of inelastic collisions with neutrals of velocity range $\vec{v}_{n_j}, d^3v_{n_j}$ such that the neutrals are placed in the set $\vec{v}_{n_k}, d^3v_{n_k}$ and the electrons are scattered through an angle $\chi_{0_{CM}}, d\Omega_{0_{CM}}$ measured relative to $\vec{v} - \vec{v}_{n_j}$ in the center-of-mass frame, with the velocity $\vec{v}_0, d^3v_0$ is

$$f_e d^3v f_{n_j} d^3v_{n_j} g_{n_j} \sigma^k_{n_j}(g_{n_j}, \chi_{0_{CM}}) d\Omega_{0_{CM}} \quad . \qquad \text{(A-4)}$$

We represent this type of transition with the following notation:

$$[\vec{v}, \vec{v}_{n_j} \rightarrow \vec{v}_0, \vec{v}_{n_k}] \quad .$$

In the previous inelastic loss expression $\sigma^k_{n_j}(g_{n_j}, \chi_{0_{CM}}) d\Omega_{0_{CM}}$ is the differential scattering cross section for the excitation collision in question. The relative velocity g_{n_j} is defined by

$$g_{n_j} \equiv |\vec{v} - \vec{v}_{n_j}| \ ,$$

$f_{n_j} \equiv f_n(\vec{v}_{n_j})$ is the velocity distribution function for the state n neutral in the state j normalized on the number density for this state.

In writing g_{n_j} and $\chi_{0_{CM}}$ as arguments of $\sigma^k_{n_j}$, we have addressed ourselves to an examination of excitation collisions which cause heavy particle state changes that can be classified by the principal and total angular momentum quantum numbers and degeneracies. The cross section should have enough information specified in its arguments to determine, along with the conservation equations, the velocities of both particles in the appropriate reference frame after the collision. When considering the dynamics of inelastic collisions in the center-of-mass coordinates, changes in the component contributions to the angular momentum vector of the system will result in the

plane of the particles after the encounter differing from the plane of the particles prior to the encounter.

From quantum mechanical considerations the uncertainty principle precludes knowledge of the precise position and velocity of a particle simultaneously. Thus, the exact value of the impact parameter, the total angular momentum vector for the system, or the plane of the particles is unknown for the collision described by (A-4). If no strong external magnetic fields exist, the direction of the total orbital angular momentum vector of the heavy particles is expected to be arbitrary. We limit our interest to the magnitude of this vector only. Then by symmetry arguments we can neglect the azimuthal dependence of the orientation of the relative velocity vector after the collision $\vec{g}_{0n_k}$ with respect to $\vec{g}_{n_j}$. Thus we only need to specify the angle $\chi_{0_{CM}}$ between $\vec{g}_{0n_k}$ and $\vec{g}_{n_j}$ to describe adequately our inelastic collisions.

This argument can also be illustrated by considering the interaction between two monoenergetic streams, one of which is taken to be the set of electrons having the velocity range $\vec{v}, d^3v$; the other stream is of neutrals in the set $\vec{v}_{n_j}, d^3v_{n_j}$. As a result of the averaging effects of the streams we expect the sum of the total angular momentum vectors of the heavy particles to be zero. This initial symmetry would be preserved after the encounter allowing us to neglect the azimuthal dependence of $\vec{g}_{0n_k}$ relative to $\vec{g}_{n_j}$. Thus, we only need to present the single deflection angle $\chi_{0_{CM}}$ as a parameter of the collision. Knowing $\chi_{0_{CM}}$, g_{n_j} and using symmetry considerations one can find the velocity of the particles after the collision in the center-of-mass reference frame.

These same arguments also apply for the other inelastic (superelastic) collisions discussed in this paper.

The number of electrons lost to the set $\vec{v}, d^3v$ in unit time per unit volume as a result of superelastic collisions described by $[\vec{v},\vec{v}_{n_k} \to \vec{v}_1,\vec{v}_{n_j}]$ is

$$f_e d^3v\, f_{n_k} d^3v_{n_k}\, g_{n_k} \sigma^j_{n_k}(g_{n_k},\chi_{1_{CM}}) d\Omega_{1_{CM}} \ . \qquad \text{(A-5)}$$

Here $\sigma^j_{n_k}(g_{n_k},\chi_{1_{CM}})d\Omega_{1_{CM}}$ is the differential cross section for this superelastic encounter. The number of electrons lost to the set per unit time per unit volume is then the sum of (A-4) and (A-5).

Similarly, the number of electrons gained by the set per unit volume per unit time as a result of the inverse encounters represented $[\vec{v}_0,\vec{v}_{n_k} \to \vec{v},\vec{v}_{n_j}]$ and $[\vec{v}_1,\vec{v}_{n_j} \to \vec{v}_1,\vec{v}_{n_k}]$ is

$$f_{e_0} d^3v_0 f_{n_k} d^3v_{n_k} g_{0n_k} \sigma^j_{n_k}(g_{0n_k},\chi_{0_{CM}}) d\Omega_{0_{CM}} \qquad \text{(A-6)}$$

and

$$f_{e_1} d^3v_1 f_{n_j} d^3v_{n_j} g_{1n_j} \sigma_{n_j}(g_{1n_j},\chi_{1_{CM}}) d\Omega_{1_{CM}} \ . \qquad \text{(A-7)}$$

Expression (A-6) is the (superelastic) inverse of (A-4). Expression (A-7) represents the inverse of (A-5). Typically the relative velocity g_{0n_k} is defined by $g_{0n_k} \equiv |\vec{v}_0 - \vec{v}_{n_k}|$.

Applying the principle of detailed balancing, we equate (A-4) and its inverse (A-6) at equilibrium and obtain

$$[f_e f_{n_j}]_{Eq}\, g_{n_j} \sigma^k_{n_j}(g_{n_j},\chi_{0_{CM}}) d^3v d^3v_{n_j} =$$

$$[f_{e_0} f_{n_k}]_{Eq}\, g_{0n_k} \sigma^j_{n_k}(g_{0n_k},\chi_{0_{CM}}) d^3v_0 d^3v_{n_k} \ . \qquad \text{(A-8)}$$

The equilibrium distribution functions are given below:

$$[f_e]_{Eq} = [n_e]_{Eq} \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-\frac{m_e v^2}{2kT}}$$

$$[f_{e_0}]_{Eq} = [n_e]_{Eq} \left(\frac{m_e}{2\pi kT}\right)^{3/2} e^{-\frac{m_e v_0^2}{2kT}}$$

$$[f_{n_k}]_{Eq} = [n_{n_k}]_{Eq} \left(\frac{m_n}{2\pi kT}\right)^{3/2} e^{-\frac{m_n v_{n_k}^2}{2kT}}$$

$$[f_{n_j}]_{Eq} = [n_{n_j}]_{Eq} \left(\frac{m_n}{2\pi kT}\right)^{3/2} e^{-\frac{m_n v_{n_j}^2}{2kT}} \qquad \text{(A-9)}$$

where T is the equilibrium temperature. At equilibrium we have $T_e = T_h = T$. The relative populations of various excited levels of an atom at equilibrium are given by the Boltzmann distribution

$$\left[\frac{n_{n_k}}{n_{n_j}}\right]_{Eq} = \frac{\omega_k}{\omega_j} e^{-\Delta_{n_{jk}}/kT} \qquad \text{(A-10)}$$

where ω_k and ω_j are the atomic degeneracies associated with the states j and k of the neutral particles. The differential velocity elements in (A-8) can be related via their Jacobian as

$$d^3v_{n_k} d^3v_0 = \left(\frac{g_{0n_k}}{g_{n_j}}\right) d^3v_{n_j} d^3v \ . \qquad \text{(A-11)}$$

Then the combination of (A-9), (A-10), (A-11), and (A-2) with (A-8) yields the following detailed balancing result:

$$g^2_{0n_k} \sigma^j_{n_k}(g_{0n_k},\chi_{0_{CM}}) = \frac{\omega_j}{\omega_k} g^2_{n_j} \sigma^k_{n_j}(g_{n_j},\chi_{0_{CM}}) \ . \qquad \text{(A-12)}$$

A similar detailed balancing analysis with (A-5) and (A-7) yields

$$g^2_{n_k} \sigma^j_{n_k}(g_{n_k},\chi_{1_{CM}}) = \frac{\omega_j}{\omega_k} g^2_{1n_j} \sigma^k_{n_j}(g_{1n_j},\chi_{1_{CM}}) . \qquad \text{(A-13)}$$

The differential velocity elements for this encounter are related by

$$d^3v_{n_k} d^3v = \left(\frac{g_{n_k}}{g_{1n_j}}\right) d^3v_{n_j} d^3v_1 \ . \qquad \text{(A-14)}$$

If the electron velocities before a superelastic and an inelastic encounter are denoted by $\vec{v}_S$ and $\vec{v}_I$ respectively, it is easy to show that Eqs. (A-12) and (A-13), relating the differential cross sections, can both be written as the following general detailed balancing result:

$$g^2_{Sn_k}\sigma^j_n(g_{Sn_k},\chi_{CM}) = \frac{\omega_j}{\omega_k} g^2_{In_j}\sigma^k_{n_j}(g_{In_j},\chi_{CM}) \ . \qquad \text{(A-15)}$$

In (A-15) g_{Sn_k} and g_{In_j} are defined as

$$g_{Sn_k} \equiv |\vec{v}_S - \vec{v}_{n_k}| \ , \text{ and}$$

$$g_{In_j} = |\vec{v}_I - \vec{v}_{n_j}| \ .$$

The velocity elements in this general notation are related by

$$d^3v_{n_k} d^3v_S = \frac{g_{Sn_k}}{g_{In_j}} d^3v_{n_j} d^3v_I \ . \qquad \text{(A-16)}$$

Equations (A-11, 12, 13, and 14) can be combined with (A-4, 5, 6, and 7) to yield an expression for the net number of electrons gained by the set $\vec{v},d^3v$ by the reactions represented by $[\vec{v}_0,\vec{v}_{n_k} \leftrightarrow \vec{v},\vec{v}_{n_j}]$ and $[\vec{v}_1,\vec{v}_{n_j} \leftrightarrow \vec{v},\vec{v}_{n_k}]$. When this resulting expression is then integrated over all possible scattering angles and heavy particle velocities, we obtain the following expression for the net gain of electrons to the set per unit volume per unit time for the reaction given by (A-1):

$$\left(\frac{\partial_e f_e}{\partial t}\right)_{\substack{Ex\\ j\leftrightarrow k}} d^3v = d^3v\Big\{\iint(f_{e_0}f_{n_k}\frac{\omega_j}{\omega_k} - f_e f_{n_j})g_{n_j}\sigma^k_{n_j}(g_{n_j},\chi_{0_{CM}})d\Omega_{0_{CM}}d^3v_{n_j}$$
$$+\iint(f_{e_1}f_{n_j}\frac{\omega_k}{\omega_j} - f_e f_{n_k})g_{n_k}\sigma^j_{n_k}(g_{n_k},\chi_{1_{CM}})d\Omega_{1_{CM}}d^3v_{n_k}\Big\} . \qquad \text{(A-17)}$$

On summing over the various states of the neutral species and dividing by d^3v we obtain

$$\left(\frac{\partial_e f_e}{\partial t}\right)_{Ex} = \sum_{k,j}\Big\{\iint(f_{e_0}f_{n_k}\frac{\omega_j}{\omega_k} - f_e f_{n_j})g_{n_j}\sigma^k_{n_j}(g_{n_j},\chi_{0_{CM}})d\Omega_{0_{CM}}d^3v_{n_j}$$
$$+\iint(f_{e_1}f_{n_j}\frac{\omega_k}{\omega_j} - f_e f_{n_k})g_{n_k}\sigma^j_{n_k}(g_{n_k},\chi_{1_{CM}})d\Omega_{1_{CM}}d^3v_{n_k}\Big\} \ . \qquad \text{(A-18)}$$

In this equation the first collision operator represents the superelastic gain and the inelastic loss to the electron set $\vec{v},d^3v$ for all possible values of j and k. The second collision operator represents the inelastic gain and the superelastic loss to this set for all possible values of j and k .

B. <u>Ionization and Three-Body Recombination Encouters</u>

In this subsection we develop collision integrals for the conservation of electrons in the set $\vec{v},d^3v$ during collisional ionization and three-body recombination encounters. A typical reaction in this case can be represented by

$$e + (\text{Atom})^j_n \rightleftarrows e + e + (\text{Ion})^\ell_n \ . \qquad \text{(A-19)}$$

Here, as previously, the subscript n corresponds to the type of neutral atom, the superscripts j and ℓ signify the state of the neutral atom and its related ion respectively.

The energy equations for this type of encounter, when we concern ourselves with how we enter or leave the set $\vec{v},d^3v$, become:

$$m_e v^2 + m_n v^2_{n_j} = m_e v^2_2 + m_e v^2_4 + m_{i_n} v^2_{i_{n\ell}} + 2\Delta_{n_{j\ell}} \qquad \text{(A-20)}$$

for the case where the ionization is caused by a $\vec{v}$ electron and

$$m_e v^2_1 + m_n v^2_{n_j} = m_e v^2 + m_e v^2_3 + m_{i_n} v^2_{i_{n\ell}} + 2\Delta_{n_{j\ell}} \qquad \text{(A-21)}$$

when the ionization results in an electron in the set. In these energy equations m_{i_n} and $\vec{v}_{i_{n\ell}}$ are the mass and velocity respectively of the ionized type n neutral atom in the state distinguished by the subscript ℓ and $\Delta_{n_{j\ell}}$ is the ionization potential of the atom from its jth state of excitation to its related ion in the ℓth state. In (A-20) $\vec{v}_2$ and $\vec{v}_4$ are the velocities of the two electrons resulting from the ionization. Making use of the principle of indistinguishability[22] we will not distinguish between bound and unbound electrons during an ionization or three-body recombination encounter. Thus, we do not specify which electron after the ionization corresponded to the ionizing electron; we merely state that two electrons result from the ionization. Likewise in considering a recombination encounter, no attempt is made to identify which of the two free electrons becomes bound and which remains free. Similarly, in (A-21) we have the case where a $\vec{v}_1$ electron caused an ionization which resulted in two electrons, one with velocity $\vec{v}$ and the other with velocity $\vec{v}_3$. It should be noted that the $\vec{v}_1$ appearing in this subsection is not the $\vec{v}_1$ that appears in subsection A of this Appendix. They are distinguished by the type interactions being considered.

The number of electrons lost to the set $\vec{v},d^3v$ in unit time per unit volume as a result of ionization encounters with neutrals, such that the following transition occurs

$$[\vec{v},\vec{v}_{n_j} \rightarrow \vec{v}_2,\vec{v}_4,\vec{v}_{1_{n\ell}}]$$

is

$$f_e d^3v f_{n_j} d^3v_{n_j} g_{n_j} \sigma^{1_{n\ell}}_{n_j \text{ion}}(g_{n_j};\, \chi_{2_{CM}},\, \vec{v}_{42_{CM}}) d^3v_{42_{CM}} d\Omega_{2_{CM}} \quad . \quad \text{(A-22)}$$

Here $\sigma^{1_{n\ell}}_{n_j \text{ion}}(g_{n_j};\, \chi_{2_{CM}},\, \vec{v}_{42_{CM}}) d^3v_{42_{CM}} d\Omega_{2_{CM}}$ is the differential "cross section" for an ionization encounter between electrons in the set $\vec{v},d^3v$ and neutrals in the set $\vec{v}_{n_j},d^3v_{n_j}$ resulting in two electrons, their velocities given by $\vec{v}_2,d^3v_2$ and $\vec{v}_4,d^3v_4$, and an ion in the state ℓ with its velocity

range and state signified by $\vec{v}_{1_{n\ell}},d^3v_{1_{n\ell}}$. The angle $\chi_{2_{CM}}$ is measured relative to the direction of $\vec{g}_{n_j}$ in the center-of-mass frame. The velocity $\vec{v}_{42_{CM}}$ is defined by $[v_{4_{CM}},\vec{\chi}_{42_{CM}}]$, where $\vec{\chi}_{42_{CM}}$ is the direction of $\vec{v}_4$ measured relative to $\vec{v}_2$ in the center-of-mass frame.

We have again used quantum mechanical and symmetry arguments to arrive at the above form of the cross section. As presented, we have specified sufficient information to be able to determine the velocities of the particles after the encounter when symmetry considerations are included with the conservation equations and the initial velocities. There is no single "correct" way to designate a differential cross section of this complexity. For example, let us examine qualitatively for a moment the dynamics of an ionizing collision as regards a choice of parameters for the cross section. Consider the interaction of a monoenergetic beam of electrons with a monoenergetic beam of neutrals which results in ionization of the neutrals. Before the ionization encounter between an electron and a neutral particle, the two particles lie in a single plane in their center-of-mass reference frame. After the encounter, however, the resulting three particles are not restricted to move in a single plane in this reference frame. By averaging all such encounters in the beams we can still expect a symmetry of sorts about the relative velocity vector $\vec{g}_{n_j}$. That is, the resultant momentum vector of any two of the scattered particles (say the electrons) from a single collision can be found equally likely in any azimuth about $\vec{g}_{n_j}$ as will the momentum vector of the third particle. These momentum vectors, that of the above resultant and that of the third particle, are equal, antiparallel, and lie in a plane which has the same symmetry about $\vec{g}_{n_j}$ as did the particles in Section A, Appendix A, after an excitation collision. This symmetry can be represented here by allowing any of the three velocity vectors to be arbitrarily located in an axial sense with respect to $\vec{g}_{n_j}$. We choose $\vec{v}_{2_{CM}}$ to have this symmetric character and thus only specify $\chi_{2_{CM}}$ in $\sigma^{1_{n\ell}}_{n_j ion}$. The collision is then completely described by specifying the velocity of one of the remaining particles relative to $\vec{v}_{2_{CM}}$. We picked $\vec{v}_4$ for this distinction. It is important to note the arbitrariness of these choices. Thus some other choice of parameters could have been made. It may appear from the above choice that the "created" electron must have the velocity $\vec{v}_4$. This is not the case. Consistent with the principle of indistinguishability we have merely stated the velocity of one of the resulting particles without specifying which particle it was prior to the encounter.

The number of electrons lost to the set $\vec{v},d^3v$ in unit time per unit volume as a result of three-body recombination encounters with ions and other electrons such that the $[\vec{v},\vec{v}_3,\vec{v}_{1_{n\ell}} \rightarrow \vec{v}_1,\vec{v}_{n_j}]$ transition occurs is

$$f_e d^3v f_{1_{n\ell}} d^3v_{1_{n\ell}} f_{e_3} d^3v_3 g_{1_{n\ell}} g_{31_{n\ell}} \sigma^{n_j}_{31_{n\ell}rec}(g_{1_{n\ell}},g_{31_{n\ell}},\vec{\chi}_{g_3 g_{CM}};\chi_{1_{CM}}) \cdot d\Omega_{1_{CM}}[1 + \delta_{\vec{v}_3,v}] . \qquad \text{(A-23)}$$

The relative velocities are defined as

$$g_{1_{n\ell}} \equiv |\vec{v} - \vec{v}_{1_{n\ell}}| \quad \text{and} \quad g_{31_{n\ell}} \equiv |\vec{v}_3 - \vec{v}_{1_{n\ell}}| .$$

$\sigma^{n_j}_{31_{n\ell}rec}(g_{1_{n\ell}},g_{31_{n\ell}},\vec{\chi}_{g_3 g_{CM}};\chi_{1_{CM}})d\Omega_{1_{CM}}$ is the "differential cross section" for this recombination collision. After this encounter the remaining free electron is scattered into the angle $\chi_{1_{CM}},d\Omega_{1_{CM}}$ measured relative to $\vec{v}_{CM}$. The angle between $\vec{g}_{31_{n\ell}}$ and $\vec{g}_{1_{n\ell}}$ is given by $\vec{\chi}_{g_3 g_{CM}}$. Again, the choice of argument for the recombination cross section is not unique, and also we do not specify which of the two free

electrons becomes bound. The delta function $\delta_{\vec{v}_3,v}$ is included in (A-23) to account for the case when both electrons participating in the recombination encounter are in the set $\vec{v},d^3v$. This function is defined as follows:

$$\delta_{\vec{v}_3,v} = \begin{cases} 0 ; & \vec{v}_3 \neq \vec{v} \\ 1 ; & \vec{v}_3 = \vec{v} \end{cases} .$$

Thus, for any smooth finite function $K(\vec{v}_3)$ we find

$$\int K(\vec{v}_3)\, \delta_{\vec{v}_3,\vec{v}}\, d^3v_3 = 0 .$$

In an analogous manner, the number of electrons <u>gained</u> by the set per unit volume per unit time as a result of the inverse recombination $[\vec{v}_2,\vec{v}_4,\vec{v}_{1_{n\ell}} \rightarrow \vec{v},\vec{v}_{n_j}]$ and ionization $[\vec{v}_1,\vec{v}_{n_j} \rightarrow \vec{v},\vec{v}_3,\vec{v}_{1_{n\ell}}]$ interactions are respectively

$$f_{e_2} d^3v_2 f_{e_4} d^3v_4 f_{1_{n\ell}} d^3v_{1_{n\ell}} g_{21_{n\ell}} g_{41_{n\ell}} \cdot \sigma^{n_j}_{241_{n\ell}rec}(g_{21_{n\ell}},g_{41_{n\ell}},\vec{\chi}_{g_4 g_{2CM}};\chi_{2_{CM}}) d\Omega_{2_{CM}} , \qquad \text{(A-24)}$$

and

$$f_{e_1} d^3v_1 f_{n_j} d^3v_{n_j} g_{1n_j} \sigma^{1_{n\ell}}_{1n_j ion}(g_{1n_j};\chi_{1_{CM}},\vec{v}_{31_{CM}}) d\Omega_{1_{CM}} d^3v_{31_{CM}} [1+\delta_{\vec{v}_3,\vec{v}}] . \qquad \text{(A-25)}$$

As a result of the collisions being inverses, the scattering angles in (A-24) and (A-25) are the same as those for the direct collision and are labeled accordingly. Note that the angles are measured relative to velocity sets and not relative to particular particles. The delta function in (A-25) is to account for the possibility of the ionization resulting in two electrons entering the set.

Applying the principle of detailed balancing we equate (A-22) and its inverse (A-24) at equilibrium and obtain

$$[f_e f_{n_j}]_{Eq}\; g_{n_j} \sigma_{n_j ion}^{i_{n\ell}}\; d^3v_{42_{CM}}\; d^3v d^3v_{n_j} = \tag{A-26}$$

$$[f_{e_2} f_{e_4} f_{i_{n\ell}}]_{Eq} g_{2i_{n\ell}} g_{4i_{n\ell}} \sigma_{24i_{n\ell}rec}^{n_j} d^3v_2 d^3v_4 d^3v_{i_{n\ell}} .$$

At equilibrium the distribution functions are Maxwellian:

$$[f_e]_{Eq} = [n_e]_{Eq} \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}$$

$$[f_{e_2}]_{Eq} = [f_e]_{Eq}\; e^{-\beta(v_2^2 - v^2)}$$

$$[f_{e_4}]_{Eq} = [f_e]_{Eq}\; e^{-\beta(v_4^2 - v^2)}$$

$$[f_{n_j}]_{Eq} = [n_{n_k}]_{Eq} \left(\frac{m_n}{2\pi kT_h}\right)^{3/2} e^{-\frac{m_n v_{n_k}^2}{2kT_h}}$$

$$[f_{i_{n\ell}}]_{Eq} = [n_{i_{n\ell}}]_{Eq} \left(\frac{m_{i_n}}{2\pi kT_h}\right)^{3/2} e^{-m_{i_n} v_{i_{n\ell}}^2/2kT_h} . \tag{A-27}$$

The number densities of the ions, the neutrals and the electrons are related at equilibrium via the Saha equation which can be written as

$$\left[\frac{n_{i_{n\ell}} n_e}{n_{n_j}}\right]_{Eq} = \frac{m_e^3}{h^3} \left(\frac{\pi}{\beta}\right)^{3/2} \frac{2\omega_\ell}{\omega_j}\; e^{-\Delta_{n_{j\ell}}/kT_h} . \tag{A-28}$$

In (A-28) ω_j and ω_ℓ are the degeneracies for the type n atom in its jth excitation level and its related ion in the ℓth excitation level. In Eqs. (A-27) and (A-28) we have set $T_e = T_h$ at equilibrium. Combining (A-20), (A-27) and (A-28) with (A-26) we obtain the following detailed balancing result:

$$g_{2i_{n\ell}} g_{4i_{n\ell}} \sigma_{24i_{n\ell}rec}^{n_j} d^3v_2 d^3v_4 d^3v_{i_{n\ell}} =$$

$$H g_{n_j} \sigma_{n_j ion}^{i_{n\ell}} d^3v_{42_{CM}} d^3v d^3v_{n_j} \tag{A-29}$$

where

$$H \equiv \frac{h^3}{m_e^3} \frac{\omega_j}{2\omega_\ell} .$$

A similar detailed balancing analysis with (A-23) and (A-25) yields

$$g_{i_{n\ell}} g_{3i_{n\ell}} \sigma_{3i_{n\ell}rec}^{n_j} d^3v d^3v_3 d^3v_{i_{n\ell}} =$$

$$H g_{1n_j} \sigma_{1n_j ion}^{i_{n\ell}} d^3v_{31_{CM}} d^3v_1 d^3v_{n_j} . \tag{A-30}$$

As was the case for inelastic and superelastic encounters, (A-29) and (A-30) can both be included in a single expression. If we let subscript Z denote the electron causing the ionization and subscripts R and B denote the electrons resulting from the ionization, we can, consistent with previous notation, include Eqs. (A-29) and (A-30) in the following equation:

$$g_{Ri_{n\ell}} g_{Bi_{n\ell}} \sigma_{RBi_{n\ell}rec}^{n_j} d^3v_R d^3v_B d^3v_{i_{n\ell}} =$$

$$H g_{Zn_j} \sigma_{Zn_j ion}^{i_{n\ell}} d^3v_{BR_{CM}} d^3v_Z d^3v_{n_j} . \tag{A-31}$$

Since the following relations between the differential velocity elements hold:

$$d^3v_R d^3v_B d^3v_{i_{n\ell}} = d^3G\; d^3g_{Ri_{n\ell}} d^3g_{Bi_{n\ell}}$$

and

$$d^3v_Z d^3v_{n_j} = d^3G\; d^3g_{Zn_j}$$

where $\vec{G}$ is the velocity of the center of mass, (A-31) can be written as

$$g_{Ri_{n\ell}} g_{Bi_{n\ell}} \sigma_{RBi_{n\ell}rec}^{n_j} d^3g_{Ri_{n\ell}} d^3g_{Bi_{n\ell}} =$$

$$H g_{Zn_j} \sigma_{Zn_j ion}^{i_{n\ell}} d^3v_{BR_{CM}} d^3g_{Zn_j} . \tag{A-32}$$

Equations (A-29) and (A-30) can be combined with (A-22, 23, 24, and 25) to yield an expression for the net number of electrons gained by the set $\vec{v}, d^3v$ as a result of interactions represented by $[\vec{v}, \vec{v}_{n_j} \leftrightarrow \vec{v}_2, \vec{v}_4, \vec{v}_{1_{n\ell}}]$ and $[\vec{v}, \vec{v}_3, \vec{v}_{1_{n\ell}} \leftrightarrow \vec{v}_1, \vec{v}_{n_j}]$. This resulting expression can then be integrated over all possible scattering angles and heavy particle velocities to obtain

$$\left(\frac{\partial_e f_e}{\partial t}\right)_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} d^3v = d^3v\Big\{\iiint (f_{e_2} f_{e_4} f_{1_{n\ell}} H - f_e f_{n_j}) g_{n_j} \sigma^{1_{n\ell}}_{n_j\text{ion}} d\Omega_{2_{CM}} d^3v_{42_{CM}} \cdot d^3v_{n_j} + \iiint (f_{e_1} f_{n_j} H^{-1} - f_e f_{1_{n\ell}} f_{e_3}) g_{1_{n\ell}} g_{31_{n\ell}} \cdot \sigma^{n_j}_{31_{n\ell}\text{rec}} d\Omega_{1_{CM}} d^3v_3 d^3v_{1_{n\ell}}\Big\} . \qquad \text{(A-33)}$$

This equation represents the net gain of electrons to the set per unit volume per unit time for the reaction given by (A-19).

Now, on summing over the relevant excited states of the neutral species and its ion and dividing by d^3v, we obtain the following collision integral for the net gain of electrons to the set per unit volume of phase space per unit time as a result of collisional ionization and three-body recombination encounters:

$$\left(\frac{\partial_e f_e}{\partial t}\right)_{\text{Ion}} = \sum_{\ell, j}\Big\{\iiint (f_{e_2} f_{e_4} f_{1_{n\ell}} H - f_e f_{n_j}) g_{n_j} \sigma^{1_{n\ell}}_{n_j\text{ion}} d\Omega_{2_{CM}} d^3v_{42_{CM}} d^3v_{n_j} + \iiint (f_{e_1} f_{n_j} H^{-1} - f_e f_{1_{n\ell}} f_{e_3}) g_{1_{n\ell}} g_{31_{n\ell}} \sigma^{n_j}_{31_{n\ell}\text{rec}} d\Omega_{1_{CM}} \cdot d^3v_3 d^3v_{1_{n\ell}}\Big\} . \qquad \text{(A-34)}$$

Here the first collision integral represents the gain by three-body recombination and the loss by ionization to the set $\vec{v}, d^3v$; the second collision integral represents the gain by ionization and the loss by three-body recombination to the set.

C. <u>Photoionization and Two-Body Recombination Encounters</u>

The remaining nonelastic collision term is that associated with photoionization. Physically, a photon of energy $h\nu_{\ell j}$ is absorbed by an atom in the state j resulting in an ion in the state ℓ and a free electron. The reverse reaction is when a free electron and an ion combine with the emission of radiation. A typical reaction for this case can be represented by

$$e + (\text{Ion})^{\ell}_{n} \rightleftarrows (\text{Atom})^{j}_{n} + \text{photon} . \qquad \text{(A-35)}$$

In terms of the relative velocity $\vec{g}_{1_{n\ell}}$ the energy equation for this encounter is

$$h\nu_{\ell j} = \tfrac{1}{2}\mu_{1_n} g^2_{1_{n\ell}} + \Delta_{n_{j\ell}} , \qquad \text{(A-36)}$$

where μ_{1_n} is the reduced mass defined by $m_e m_{1_n}/(m_e + m_{1_n})$. We consider only nonrelativistic electrons in this analysis and neglect the momentum of the photons relative to the momentum of the electrons or the heavy particles. Then in the center-of-mass reference frame the direct encounter (photoionization) will result in an electron and an ion moving in an antiparallel direction arbitrarily oriented relative to the direction of the incoming photon. Angular momentum considerations can be used to spatially orient these resulting particles relative to the initial total orbital angular momentum vector of the neutral particle for a particular collision. If we consider the interaction between a monoenergetic photon beam and a cloud of neutrals in the state j in the center-of-mass frame which are not polarized by an external magnetic field, then we can conclude by symmetry arguments that the collision probabilities will be independent of the "scattering" angle of the recoiling particles. Thus, in any direct collision $\nu_{\ell j}$ would be the only parameter needed, in addition to the conservation equations, to determine the velocities of the resulting ion and electron in a particular direction.

Let $\psi_{n_j}^{i_{n\ell}}(\nu_{\ell j})\bar{I}(\nu_{\ell j})d\nu_{\ell j}d\Omega_{CM}$ be the probability that a neutral atom in state j and set $\vec{v}_{n_j},d^3v_{n_j}$ will in unit time, under the influence of $\nu_{\ell j}$-radiation of intensity $\bar{I}(\nu_{\ell j})d\nu_{\ell j}$, become ionized by absorption of a quantum of energy $h\nu_{\ell j}$ and emit an electron into the angle $d\Omega_{CM}$ with speed v,dv; the resulting ion will be in the state ℓ with velocity $\vec{v}_{i_{n\ell}},d^3v_{i_{n\ell}}$. The solid angle associated with the ion motion will be $-d\Omega_{CM}$ in the center-of-mass frame. $\bar{I}(\nu_{\ell j})$ is the specific intensity of $\nu_{\ell j}$-radiation integrated over all solid angles; it is equal to the product of the radiant energy density at this frequency $\rho(\nu_{\ell j})$ and the speed of light c. The number of electrons gained by the set $\vec{v},d^3v$ per unit time per unit volume as a result of photoionization can then be represented by

$$f_{n_j}d^3v_{n_j}\psi_{n_j}^{i_{n\ell}}(\nu_{\ell j})\bar{I}(\nu_{\ell j})d\nu_{\ell j}d\Omega_{CM}\ . \tag{A-37}$$

Two-body recombination, the inverse of the above encounter, results in a depletion of electrons from the set. An examination of the dynamical equations in the center-of-mass frame will reveal that the only parameter needed to determine the frequency of the radiant energy is the relative speed $g_{i_{n\ell}}$. Then, defining $\bar{P}_{i_{n\ell}}^{n_j}(g_{i_{n\ell}})$ as the differential cross section for radiative capture (two-body recombination) of an electron in the velocity range $\vec{v},d^3v$ by an ion in the set $\vec{v}_{i_{n\ell}},d^3v_{i_{n\ell}}$ which results in the emission of a quantum of radiation $h\nu_{\ell j}$ and a neutral atom in the set $\vec{v}_{n_j},d^3v_{n_j}$, we can write an expression for the number of electrons lost by radiative capture for the reaction represented by (A-35) as

$$f_e d^3v f_{i_{n\ell}} d^3v_{i_{n\ell}} g_{i_{n\ell}} \bar{P}_{i_{n\ell}}^{n_j}(g_{i_{n\ell}})\ . \tag{A-38}$$

Electron captures are both spontaneous and stimulated. Thus $\bar{P}_{i_{n\ell}}^{n_j}$ can be written as

$$\bar{P}_{i_{n\ell}}^{n_j}(g_{i_{n\ell}}) = 4\pi\alpha_{i_{n\ell}}^{n_j}(g_{i_{n\ell}}) + \bar{I}(\nu_{\ell j})\beta_{i_{n\ell}}^{n_j}(g_{i_{n\ell}}) \tag{A-39}$$

where the first term on the right-hand side is the contribution from spontaneous capture and the remaining term is the contribution from induced capture.

Applying the principle of detailed balancing we can equate (A-37) and (A-38) at equilibrium. Utilizing (A-39) with this equality we obtain

$$[f_e f_{i_{n\ell}}]_{Eq}\, g_{i_{n\ell}}\left\{4\pi\alpha_{i_{n\ell}}^{n_j}+[\bar{I}(\nu_{\ell j})]_{Eq}\,\beta_{i_{n\ell}}^{n_j}\right\} d^3v d^3v_{i_{n\ell}} =$$
$$[f_{n_j}]_{Eq}\,\psi_{n_j}^{i_{n\ell}}(\nu_{\ell j})[\bar{I}(\nu_{\ell j})]_{Eq}d^3v_{n_j}d\nu_{\ell j}d\Omega_{CM}\ . \tag{A-40}$$

At equilibrium the distribution functions are Maxwellian and are given by (A-27). The equilibrium specific radiation intensity, given below, is Planck's black body intensity:

$$[\bar{I}(\nu_{\ell j})]_{Eq} = \frac{8\pi h\nu_{\ell j}^3}{c^2}\left(\frac{1}{e^{h\nu_{\ell j}/kT_h}-1}\right)\ . \tag{A-41}$$

The Saha equation (A-28) relates the number densities in (A-40). Combining (A-27), (A-28), and (A-41) with (A-40), we obtain

$$H^{-1}g_{i_{n\ell}}[\alpha_{i_{n\ell}}^{n_j}(1-e^{-h\nu_{\ell j}/kT_h})+(\frac{2h\nu_{\ell j}^3}{c^2})e^{-h\nu_{\ell j}/kT_h}\beta_{i_{n\ell}}^{n_j}]d^3vd^3v_{i_{n\ell}} =$$
$$\psi_{n_j}^{i_{n\ell}}(\nu_{\ell j})\frac{2h\nu_{\ell j}^3}{c^2}\,d^3v_{n_j}d\nu_{\ell j}d\Omega_{CM}\ . \tag{A-42}$$

Since this equation must be independent of the temperature the following relations must hold:

$$\alpha_{i_{n\ell}}^{n_j}(g_{i_{n\ell}}) = \frac{2h\nu_{\ell j}^3}{c^2}\,\beta_{i_{n\ell}}^{n_j}(g_{i_{n\ell}})\ , \quad \text{and} \tag{A-43}$$

$$\psi_{n_j}^{i_{n\ell}}(\nu_{\ell j})d\Omega_{CM}d\nu_{\ell j}d^3v_{n_j} = H^{-1}\frac{c^2}{2h\nu_{\ell j}^3}\,g_{i_{n\ell}}\alpha_{i_{n\ell}}^{n_j}(g_{i_{n\ell}})d^3vd^3v_{i_{n\ell}}\ . \tag{A-44}$$

The differential elements in (A-44) can be related through the Jacobian of the following transformation:

$$d\Omega_{CM}\, d\nu_{\ell j}\, d^3v_{n_j} = |J|\, d^3v\, d^3v_{i_{n\ell}} \qquad \text{(A-45)}$$

To evaluate $|J|$ we make use of the following differential relationship which holds for our photoionizing model:

$$d^3v\, d^3v_{i_{n\ell}} = d^3g_{i_{n\ell}}\, d^3v_{n_j} .$$

When this relation is combined with (A-45) the result can be reduced to

$$d\Omega_{CM}\, d\nu_{\ell j} = |J|\, d^3g_{i_{n\ell}} . \qquad \text{(A-46)}$$

Since $d^3g_{i_{n\ell}} \equiv g^2_{i_{n\ell}}\, dg_{i_{n\ell}}\, d\Omega_{CM}$, we can write (A-46) as

$$d\nu_{\ell j} = |J|\, g^2_{i_{n\ell}}\, dg_{i_{n\ell}}$$

so that

$$|J| = \frac{1}{g^2_{i_{n\ell}}} \frac{\partial \nu_{\ell j}}{\partial g_{i_{n\ell}}} .$$

Then using (A-36) we can find the following relationship

$$\frac{\partial \nu_{\ell j}}{\partial g_{i_{n\ell}}} = \frac{\mu_{i_n} g_{i_{n\ell}}}{h} .$$

Thus the Jacobian becomes

$$|J| = \frac{\mu_{i_n}}{h g_{i_{n\ell}}}$$

and (A-45) can then be written as

$$d\Omega_{CM}\, d\nu_{\ell j}\, d^3v_{n_j} = \frac{\mu_{i_n}}{h g_{i_{n\ell}}}\, d^3v\, d^3v_{i_{n\ell}} . \qquad \text{(A-47)}$$

With (A-47) and the definition of H , Eq. (A-44) can be reduced to

$$\psi^{i_{n\ell}}_{n_j}(\nu_{\ell j}) = \frac{m_e c^2}{h^3 \nu^3_{\ell j}} \left(\frac{2\omega_\ell}{\omega_j} \right) \left(\frac{m_e g^2_{i_{n\ell}}}{2} \right) \alpha^{n_j}_{i_{n\ell}}(g_{i_{n\ell}}) \qquad \text{(A-48)}$$

where we have replaced μ_{i_n} by m_e . The variable $g_{i_{n\ell}}$ on

the right-hand side of (A-48) is related to $\nu_{\ell j}$ by Eq. (A-36). Equations (A-43) and (A-48) are the detailed balancing relations for this photoionization encounter.

When these detailed balance expressions are introduced into the gain expression (A-37) and the result combined with the loss expression (A-39) and then integrated over all possible ion velocities we obtain

$$\left(\frac{\partial_e f_e}{\partial t}\right)_{\substack{Ph \\ j\leftrightarrow\ell}} d^3v = d^3v \int \left[f_{n_j} H^{-1} \frac{I\beta^{n_j}_{i_{n\ell}}}{P^{n_j}_{i_{n\ell}}} - f_e f_{i_{n\ell}} \right] g_{i_{n\ell}} P^{n_j}_{i_{n\ell}}\, d^3v_{i_{n\ell}} . \qquad \text{(A-49)}$$

This equation represents the net gain of electrons to the set $\vec{v}, d^3v$ per unit volume per unit time for the reaction represented by (A-35). On summing over all possible states we obtain the following collision integral for photoionization:

$$\left(\frac{\partial_e f_e}{\partial t}\right)_{Ph} = \sum_{\ell, j} \int \left[f_{n_j} H^{-1} \frac{I\beta^{n_j}_{i_{n\ell}}}{P^{n_j}_{i_{n\ell}}} - f_e f_{i_{n\ell}} \right] g_{i_{n\ell}} P^{n_j}_{i_{n\ell}}\, d^3v_{i_{n\ell}} . \qquad \text{(A-50)}$$

II. The Lorentz Approximation

In a uniform applied field the distribution function f_e will exhibit azimuthal symmetry about $\vec{E}$. We take advantage of this symmetry and employ the truncated expansion of f_e in azimuthally symmetric spherical harmonics[4], otherwise known as the Lorentz approximation[2,8,23]. That is, we assume $f_e(\vec{v},t)$ can be accurately represented by

$$f_e(\vec{v},t) = f(\mu,v,t) = \sum_m f^m(v,t) P_m(\mu) \simeq f^0(v,t) + \mu f^1(v,t) \qquad \text{(A-51)}$$

where the P_m are Legendre polynomials and μ is the cosine of the angle between $\vec{v}$ and the direction of the electric field. The Lorentz approximation is valid for small deviations from spherical symmetry, in which case $f^1 \ll f^0$ and the remaining terms in the expansion are negligible.

A. Application to Equation (1)

Working with spherical coordinates, we combine (A-51) with the terms in Eq. (1), utilize the simplifications consistent with the small electron mass and the orthogonality property of Legendre polynomials, and obtain as a result the following coupled equations for f^0 and f^1 :

$$\frac{\partial f^0}{\partial t} + \frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{\gamma}{3} v^2 f^1 - \frac{v^4}{2\beta}\frac{\delta}{\lambda_E}\left(\frac{1}{2v}\frac{\partial f^0}{\partial v} + \beta f^0\right) - f^0 I^{0,v}_{0,0} - \frac{v}{3}\frac{\partial f^0}{\partial v}\left(I^{0,v}_{2,0} + I^{0,\infty}_{-1,v}\right)\right\} = \left(\frac{\partial_e f^0}{\partial t}\right)_{NE} , \quad \text{(A-52)}$$

$$\frac{\partial f^1}{\partial t} + \gamma\frac{\partial f^0}{\partial v} = -\frac{v f^1}{\gamma_E} + \left(\frac{\partial_e f^1}{\partial t}\right)_e + \left(\frac{\partial_e f^1}{\partial t}\right)_{NE} . \quad \text{(A-53)}$$

The first two terms in both these equations are the direct result of applying the Lorentz approximation to the left-hand side of (1). The second term in the brackets in (A-52) results from the isotropic part of the elastic electron-heavy particle (ion and neutral) collision terms[4,5]. δ , an effective mass ratio, is defined as follows:

$$\delta \equiv \lambda_E\left[\sum_n \frac{\delta_n}{\lambda_n} + \sum_i \frac{\delta_i}{\lambda_i}\right]^* , \quad \text{(A-54)}$$

where $\delta_n = \frac{2m_e}{m_n}$, $\delta_i = \frac{2m_e}{m_i}$, and λ_E , the mean free path for elastic momentum transfer between electrons and heavy species, is defined by

$$\frac{1}{\lambda_E} \equiv \sum_n \frac{1}{\lambda_n} + \sum_i \frac{1}{\lambda_i} . \quad \text{(A-55)}$$

The mean free paths for momentum transfer between the electrons and the neutral and ion species, λ_n and λ_i respectively, are given by

$$\frac{1}{\lambda_n} = n_n \int (1-\cos\chi)\sigma_n(\chi,g_n)d\Omega , \quad \text{(A-56)}$$

*As defined, δ is nearly independent of velocity.

and

$$\frac{1}{\lambda_i} = n_i\, 2\pi b_0^2\, \ell n\left[1 + \left(\frac{\lambda_D}{b_0}\right)^2\right] \quad \text{(A-57)}$$

where

$\lambda_D = \left(\frac{kT_e}{4\pi n_e e^2}\right)^{1/2}$ is the Debye length,

T_e is the electron temperature,

$b_0 = \left(\frac{e^2}{\mu_i v^2}\right)$ is the impact parameter for a 90° Coulomb deflection,

$\mu_i = \frac{m_e m_i}{m_e + m_i}$ is the reduced mass.

The terms containing I^{q,v_2}_{p,v_1} correspond to the isotropic part of the electron-electron iteractions[5]. I^{q,v_2}_{p,v_1} is defined by

$$I^{q,v_2}_{p,v_1} \equiv \frac{4\pi\Gamma_{ee}}{v^p}\int_{v_1}^{v_2} f^q v^{2+p} dv . \quad \text{(A-58)}$$

On the right-hand side of (A-52) is the contribution from all the nonelastic collisions. This term will be treated in some detail later in Section II,B of this appendix.

The first term on the right-hand side of (A-53) represents the anisotropic collisional contributions to the kinetic equation as a result of elastic electron-heavy particle encounters[5]. The second term on the right-hand side of (A-53) is the anisotropic contribution of the electron-electron interactions. Written out explicitly, it is[5]

$$\left(\frac{\partial_e f^1}{\partial t}\right)_e = 8\pi\Gamma_{ee} f^0 f^1 + \frac{1}{15v^2}\frac{df^0}{dv}\left(-3I^{1,v}_{3,0} + 2I^{1,\infty}_{-2,v} + 5I^{1,v}_{1,0}\right) + \frac{1}{5v}\frac{d^2f^0}{dv^2}\left(I^{1,v}_{3,0} + I^{1,\infty}_{-2,v}\right) + \frac{1}{3v}\frac{d^2f^1}{dv^2}\left(I^{0,v}_{2,0} + I^{0,\infty}_{-1,v}\right) + \frac{1}{3v^2}\left(\frac{df^1}{dv} - \frac{f^1}{v}\right)\left(3I^{0,v}_{0,0} - I^{0,v}_{2,0} + 2I^{0,\infty}_{-1,v}\right) . \quad \text{(A-59)}$$

The remaining term in (A-53) represents the anisotropic contribution to Eq. (1) as a result of all nonelastic collisions. This term, along with $(\partial_e f^0/\partial t)_{NE}$, will be developed below.

B. Application to Nonelastic Collision Integrals

It follows from the substitution of (A-11) into Eq. (1) and the subsequent operations which led to Eqs. (A-52) and (A-53) that the nonelastic terms in this latter set of equations can be written as

$$(\frac{\partial_e f^0}{\partial t})_{NE} = \sum_n (\frac{\partial_e f^0}{\partial t})_{Ex} + \sum_n (\frac{\partial_e f^0}{\partial t})_{Ion} + \sum_n (\frac{\partial_e f^0}{\partial t})_{Ph} \quad \text{(A-60)}$$

and

$$(\frac{\partial_e f^1}{\partial t})_{NE} = \sum_n (\frac{\partial_e f^1}{\partial t})_{Ex} + \sum_n (\frac{\partial_e f^1}{\partial t})_{Ion} + \sum_n (\frac{\partial_e f^1}{\partial t})_{Ph} \,. \quad \text{(A-61)}$$

In this section we will illustrate how the terms on the right-hand sides of (A-60) and (A-61) can be reduced to obtain

$$(\frac{\partial_e f^0}{\partial t})_{NE} = (F_{NE} - f^0)\frac{v}{\lambda^0_{NE}} \quad \text{(A-62)}$$

and

$$(\frac{\partial_e f^1}{\partial t})_{NE} = -\frac{v f^1}{\lambda_{NE}} \,. \quad \text{(A-63)}$$

In (A-62) F_{NE} represents a gain of electrons to the set $\vec{v}, d^3v$ via nonelastic encounters and the term proportional to f^0 corresponds to a loss from this set. F_{NE} and the effective mean free paths λ_{NE} and λ^0_{NE} associated with nonelastic collisions will be defined in the paragraphs to follow.

We begin by considering each type of elastic encounter separately:

1. Inelastic and Superelastic Collisions

On the substitution of (A-51) into a typical

term in (A-18), for example (A-17), and separating the result into even and odd powers of μ , we find

$$(\frac{\partial_e f^0}{\partial t})_{\substack{Ex\\ j\leftrightarrow k}} = \Big\{\iint(f^0_0\, f_{n_k}\frac{\omega_j}{\omega_k} - f^0 f_{n_j})g_{n_j}\sigma^k_{n_j}\,d\Omega_{0_{CM}}\,d^3v_{n_j} + \iint(f^0_1\, f_{n_j}\frac{\omega_k}{\omega_j} - f^0 f_{n_k})g_{n_k}\sigma^j_{n_k}\,d\Omega_{1_{CM}}\,d^3v_{n_k}\Big\} \quad \text{(A-64)}$$

and

$$(\frac{\partial_e \mu f^1}{\partial t})_{\substack{Ex\\ j\leftrightarrow k}} = \Big\{\iint(\mu_0 f^1_0\, f_{n_k}\frac{\omega_j}{\omega_k} - \mu f^1 f_{n_j})g_{n_j}\sigma^k_{n_j}\,d\Omega_{0_{CM}}\,d^3v_{n_j} + \iint(\mu_1 f^1_1\, f_{n_j}\frac{\omega_k}{\omega_j} - \mu f^1 f_{n_k})g_{n_k}\sigma^j_{n_k}\,d\Omega_{1_{CM}}\,d^3v_{n_k}\Big\} \,. \quad \text{(A-65)}$$

Utilizing the small electron mass approximation $m_e/m_h \sim 0$ here, we can neglect changes in the heavy particle velocity during these encounters. Thus in Eqs. (A-64) and (A-65) differential velocity elements for the heavy particles can be interchanged. Also, since for the most part the electron speeds are so much greater than the speed of the heavy particles, we can to this approximation replace the relative speeds in these equations by the appropriate electron speeds.

Based on these approximations the following relationships can be shown to exist between the angles which describe the direction of the electron motion relative to the direction of the applied field:

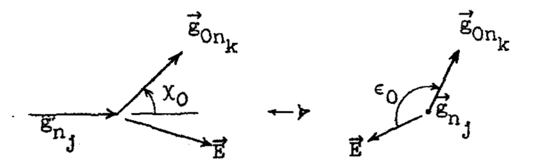
$$\mu_0 = \mu\cos\chi_{0_{CM}} + \sqrt{1-\mu^2}\,\sin\chi_{0_{CM}}\cos\epsilon_{0_{CM}}$$

and

$$\mu_1 = \mu\cos\chi_{1_{CM}} + \sqrt{1-\mu^2}\,\sin\chi_{1_{CM}}\cos\epsilon_{1_{CM}} \,. \quad \text{(A-66)}$$

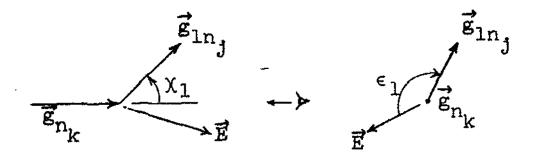
The sketch below showing the relative velocities before and after a collision will help clarify these relationships.

Inelastic loss: (center-of-mass frame)



$\vec{g}_{n_j}$ in plane of page $\quad$ $\vec{g}_{n_j}$ normal to page

Superelastic loss: (center-of-mass frame)



$\vec{g}_{n_k}$ in plane of page $\quad$ $\vec{g}_{n_k}$ normal to page

Once the small mass approximation has been utilized it is no longer necessary to differentiate between center-of-mass and laboratory coordinate frames in regard to electron motion. Thus, in (A-66) and subsequent equations in this subsection the CM subscript could be deleted.

When (A-66) and the results of the small mass approximation are combined with (A-64) and (A-65) and the integration over the heavy particle velocity is performed, we obtain

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{\substack{Ex \\ j\leftrightarrow k}} = \left(\frac{n_{n_k}}{n_{n_j}}\frac{\omega_j}{\omega_k} f_0^0 - f^0\right)\frac{v}{\lambda_{In_j}} + \left(\frac{n_{n_j}}{n_{n_k}}\frac{\omega_k}{\omega_j} f_1^0 - f^0\right)\frac{v}{\lambda_{In_k}} \qquad \text{(A-67)}$$

and

$$\left(\frac{\partial_e f^1}{\partial t}\right)_{\substack{Ex \\ j\leftrightarrow k}} = -\frac{v f^1}{\lambda_{IS_{jk}}} . \qquad \text{(A-68)}$$

In Eq. (A-67) the following relations and definitions hold:

$$f_0^0 = f^0\left(\sqrt{v^2 - \frac{2\Delta_{n_{jk}}}{\mu_n}}\right) ,$$

$$f_1^0 = f^0\left(\sqrt{v^2 + \frac{2\Delta_{n_{jk}}}{\mu_n}}\right) , \quad (\mu_n \equiv \text{reduced mass})$$

$$\frac{1}{\lambda_{In_j}} \equiv n_{n_j} \int \sigma_{n_j}^k \, d\Omega_{0_{CM}} ,$$

and

$$\frac{1}{\lambda_{In_k}} \equiv n_{n_k} \int \sigma_{n_k}^j \, d\Omega_{1_{CM}} .$$

The arguments of the terms f_0^0 and f_1^0 used here are not the same as the arguments of similarly labeled terms that will appear later in connection with collisional ionization and its inverse. The context in which these distribution functions are used should serve to differentiate their arguments.

In (A-68) $\lambda_{IS_{jk}}$ is the effective mean free path for momentum transfer as a result of inelastic and superelastic collisions; it is defined by

$$\frac{1}{\lambda_{IS_{jk}}} \equiv \frac{1}{\lambda_{In_j}} + \frac{1}{\lambda_{In_k}} - \frac{1}{\lambda_{IS_0}} - \frac{1}{\lambda_{IS_1}} \qquad \text{(A-69)}$$

where

$$\frac{1}{\lambda_{IS_0}} \equiv \frac{f_0^1}{f^1}\left(n_{n_k}\frac{\omega_j}{\omega_k}\int \cos\chi_{0_{CM}}\,\sigma_{n_j}^k \, d\Omega_{0_{CM}}\right)$$

and

$$\frac{1}{\lambda_{IS_1}} \equiv \frac{f_1^1}{f^1}\left(n_{n_j}\frac{\omega_k}{\omega_j}\int \cos\chi_{1_{CM}}\,\sigma_{n_k}^j \, d\Omega_{1_{CM}}\right) .$$

On summing (A-67) and (A-68) over the various states of the neutral species we obtain finally

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{Ex} = \sum_{j,k}\left\{\left(\frac{n_{n_k}}{n_{n_j}}\frac{\omega_j}{\omega_k} f_0^0 - f^0\right)\frac{v}{\lambda_{In_j}} + \left(\frac{n_{n_j}}{n_{n_k}}\frac{\omega_k}{\omega_j} f_1^0 - f^0\right)\frac{v}{\lambda_{In_k}}\right\} \qquad \text{(A-70)}$$

and

$$\left(\frac{\partial_e f^1}{\partial t}\right)_{Ex} = -vf^1 \sum_{j,k} \frac{1}{\lambda_{IS_{jk}}} . \tag{A-71}$$

2. Ionization and Three-Body Recombination Encounters

Introducing the Lorentz approximation (A-51) for the electron distribution function into a typical term in (A-34), say (A-33), and separating the result into even and odd powers of μ, we obtain

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{\substack{Ion \\ j\leftrightarrow\ell}} = \Big\{ \iiint (f_2^0 f_4^0 f_{i_{n\ell}} H - f^0 f_{n_j}) g_{n_j} \sigma_{n_j\,ion}^{i_{n\ell}} d\Omega_{2_{CM}} d^3v_{42_{CM}} d^3v_{n_j}$$

$$+ \iiint (f_1^0 f_{n_j} H^{-1} - f^0 f_{i_{n\ell}} f_3^0) g_{i_{n\ell}} g_{3i_{n\ell}} \sigma_{3i_{n\ell}\,rec}^{n_j} d\Omega_{1_{CM}} d^3v_3 d^3v_{i_{n\ell}} \Big\} \tag{A-72}$$

and

$$\left(\frac{\partial_e \mu f^1}{\partial t}\right)_{\substack{Ion \\ j\leftrightarrow\ell}} = \Big\{ \iiint [(\mu_2 f_2^1 f_4^0 + \mu_4 f_2^0 f_4^1) f_{i_{n\ell}} H - \mu f^1 f_{n_j}] g_{n_j} \sigma_{n_j\,ion}^{i_{n\ell}} d\Omega_{2_{CM}}$$

$$\cdot d^3v_{42_{CM}} d^3v_{n_j} + \iiint [\mu_1 f_1^1 f_{n_j} H^{-1} - f_{i_{n\ell}} (\mu f^1 f_3^0 + \mu_3 f^0 f_3^1)]$$

$$\cdot g_{i_{n\ell}} g_{3i_{n\ell}} \sigma_{3i_{n\ell}\,rec}^{n_j} d\Omega_{1_{CM}} d^3v_3 d^3v_{i_{n\ell}} \Big\} . \tag{A-73}$$

In (A-72) we have neglected terms which contain the product of two anisotropic parts of the electron distribution function. This is justified on the basis of f^1 being very much less than f^0 for small deviations from spherical symmetry.

Now, following the treatment of (A-64) and (A-65), we introduce the small mass approximation into the above equations and obtain after integrating over the velocity of the heavy particles:

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{\substack{Ion \\ j\leftrightarrow\ell}} = \Big\{ \iint (f_2^0 f_4^0 n_{i_{n\ell}} H - n_{n_j} f^0) v \sigma_{n_j\,ion}^{i_{n\ell}} d\Omega_{2_{CM}} d^3v_{42_{CM}}$$

$$+ \iint (f_1^0 n_{n_j} H^{-1} - n_{i_{n\ell}} f^0 f_3^0) v v_3 \sigma_{3i_{n\ell}\,rec}^{n_j} d\Omega_{1_{CM}} d^3v_3 \Big\} \tag{A-74}$$

and

$$\left(\frac{\partial_e \mu f^1}{\partial t}\right)_{\substack{Ion \\ j\leftrightarrow\ell}} = \Big\{ \iint [\mu_2 f_2^1 f_4^0 + \mu_4 f_2^0 f_4^1) n_{i_{n\ell}} H - n_{n_j} \mu f^1] v \sigma_{n_j\,ion}^{i_{n\ell}} d\Omega_{2_{CM}} d^3v_{42_{CM}}$$

$$+ \iint [\mu_1 f_1^1 n_{n_j} H^{-1} - n_{i_{n\ell}} (\mu f^1 f_3^0 + \mu_3 f^0 f_3^1)] v v_3 \sigma_{3i_{n\ell}\,rec}^{n_j}$$

$$\cdot d\Omega_{1_{CM}} d^3v_3 \Big\} . \tag{A-75}$$

For these encounters, the following angular relationships can be shown to exist:

$$\mu_2 = \mu \cos \chi_{2_{CM}} + \sqrt{1-\mu^2} \sin \chi_{2_{CM}} \cos \epsilon_{2_{CM}} ,$$

$$\mu_4 = \mu \cos \chi_{4_{CM}} + \sqrt{1-\mu^2} \sin \chi_{4_{CM}} \cos \epsilon_{4_{CM}} ,$$

$$\mu_1 = \mu \cos \chi_{1_{CM}} + \sqrt{1-\mu^2} \sin \chi_{1_{CM}} \cos \epsilon_{1_{CM}} ,$$

and

$$\mu_3 = \mu \cos \chi_{3_{CM}} + \sqrt{1-\mu^2} \sin \chi_{3_{CM}} \cos \epsilon_{3_{CM}} . \tag{A-76}$$

The μ_r variables on the left-hand side of the above expressions represent the cosine of the co-latitude angles subtended by the electron velocities $\vec{v}_r$ and the electric field. The scattering angles are all measured relative to the velocity $\vec{v}$ (thus, here $\chi_{3_{CM}} \equiv \chi_{v_3 v_{CM}}$). This velocity vector also acts as the polar axis for the azimuthal angles ϵ_r which are measured from the plane containing $\vec{v}$ and the electric field. When (A-76) is combined with (A-75) the $\cos \epsilon_{1_{CM}}$ and $\cos \epsilon_{2_{CM}}$ terms will not contribute when the integration over these azimuthal angles is performed. The azimuthal orientation of the specific angles $\epsilon_{3_{CM}}$ and $\epsilon_{4_{CM}}$ is not important even though differences between these and other angles appear in the cross section arguments. Thus, the integration over $d\epsilon_3$ ($d\epsilon_3 = d\epsilon_{3_{CM}}$ for small mass approximation) will result in no contribution from the $\cos \epsilon_{3_{CM}}$ term, and since integration over $d\Omega_{2_{CM}}$ and $d\Omega_{42_{CM}}$ is equivalent to an integration over $d\Omega_{4_{CM}}$ we find the $\cos \epsilon_{4_{CM}}$ term also will not contribute when these integrations are performed.

It should be noted again that once the small mass approximation has been made (here as well as in the previous subsection) it is no longer necessary to continue using the center-of-mass subscript notation. Nevertheless, in order not to lose sight of the various coordinate frames we are utilizing we will continue to use this notation, keeping in mind, however, that in regard to electron motion the center of mass is essentially at rest so that to this approximation electron velocities can be regarded as equal in either the laboratory or center-of-mass reference frames. Thus, in the previous paragraph we were able to treat $d\epsilon_3$ as $d\epsilon_{3_{CM}}$.

Equations (A-74) and (A-75) can now be written in the following abbreviated form:

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} = (F_{I_{j\ell}} - f^0)\frac{v}{\lambda_{\text{Ion}_{j\ell}}} + (F_{R_{\ell j}} - f^0)\frac{v}{\lambda_{\text{Rec}_{\ell j}}} \tag{A-77}$$

and

$$\left(\frac{\partial_e f^1}{\partial t}\right)_{\substack{\text{Ion}\\ j\leftrightarrow\ell}} = -\frac{vf^1}{\lambda_{\text{IR}_{j\ell}}} . \tag{A-78}$$

In (A-77) the following definitions are used:

$$\frac{F_{I_{j\ell}}(v)}{\lambda_{\text{Ion}_{j\ell}}} \equiv n_{1_{n\ell}} H \iiint_0^{v_{4_{max}}} f_2^0 f_4^0 \sigma_{n_j\text{ion}}^{1_{n\ell}} d\Omega_{2_{CM}} d\Omega_{42_{CM}} v_4^2 dv_4$$

where $v_4 = v_{4_{CM}}$ in the small mass approximation

$$v_2 = \sqrt{v^2 - v_4^2 - \frac{2}{\mu_n}\Delta_{n_{j\ell}}} \quad \text{from (A-20)} ,$$

$$\frac{F_{R_{\ell j}}(v)}{\lambda_{\text{Rec}_{\ell j}}} \equiv n_{n_j} H^{-1} \iiint_0^{v_{3_{max}}} f_1^0 v_3 \sigma_{3 1_{n\ell}\text{rec}}^{n_j} d\Omega_{1_{CM}} d\Omega_3 v_3^2 dv_3$$

where $v_1 = \sqrt{v^2 + v_3^2 + \frac{2}{\mu_n}\Delta_{n_{j\ell}}}$ from (A-21) ,

$$\frac{1}{\lambda_{\text{Ion}_{j\ell}}} \equiv n_{n_j} \langle \sigma_{n_j\text{ion}}^{1_{n\ell}} \rangle$$

where

$$\langle \sigma_{n_j\text{ion}}^{1_{n\ell}} \rangle \equiv \iiint_0^{v_{4_{max}}} \sigma_{n_j\text{ion}}^{1_{n\ell}} d\Omega_{2_{CM}} d\Omega_{42_{CM}} v_4^2 dv_4$$

is the mean effective area for ionization, and

$$\frac{1}{\lambda_{\text{Rec}_{\ell j}}} \equiv n_{1_{n\ell}} \langle \sigma_{3 1_{n\ell}\text{rec}}^{n_j} \rangle$$

where

$$\langle \sigma_{3 1_{n\ell}\text{rec}}^{n_j} \rangle \equiv \iiint_0^{v_{3_{max}}} v_3^3 f_3^0 \, \sigma_{3 1_{n\ell}\text{rec}}^{n_j} d\Omega_{1_{CM}} d\Omega_3 dv_3$$

is the "mean effective area" for recombination.

In (A-78) $\lambda_{\text{IR}_{j\ell}}$ is the effective mean free path for momentum transfer caused by ionization and recombination interactions. It is defined by

$$\frac{1}{\lambda_{\text{IR}_{j\ell}}} \equiv -\frac{1}{\lambda_{\text{I2}_{j\ell}}} + \frac{1}{\lambda_{\text{Ion}_{j\ell}}} - \frac{1}{\lambda_{\text{R1}_{\ell j}}} + \frac{1}{\lambda_{\text{Rec}_{\ell j}}} + \frac{1}{\lambda_{R_{\ell j}}} \tag{A-79}$$

where

$$\frac{1}{\lambda_{\text{I2}_{j\ell}}} \equiv n_{1_{n\ell}} \frac{H}{f^1} \iiint_0^{v_{4_{max}}} (f_2^1 f_4^0 \cos\chi_{2_{CM}} + f_2^0 f_4^1 \cos\chi_{4_{CM}}) \sigma_{n_j\text{ion}}^{1_{n\ell}} d\Omega_{2_{CM}} d\Omega_{42_{CM}} v_4^2 dv_4 ,$$

$$\frac{1}{\lambda_{\text{R1}_{\ell j}}} \equiv \frac{n_{n_j} H^{-1}}{f^1} \iiint_0^{v_{3_{max}}} f_1^1 \cos\chi_{1_{CM}} v_3^3 \, \sigma_{3 1_{n\ell}\text{rec}}^{n_j} d\Omega_{1_{CM}} d\Omega_3 dv_3 ,$$

and

$$\frac{1}{\lambda_{R_{\ell j}}} \equiv \frac{n_{1_{n\ell}}}{f^1} \iiint_0^{v_{3_{max}}} f^0 f_3^1 \cos\chi_{3_{CM}} v_3^3 \, \sigma_{3 1_{n\ell}\text{rec}}^{n_j} d\Omega_{1_{CM}} d\Omega_3 dv_3 .$$

On summing (A-77) and (A-78) over the various states, we finally obtain:

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{Ion} = \sum_{\ell,j} \left\{ (F_{I_{j\ell}} - f^0)\frac{v}{\lambda_{Ion_{j\ell}}} + (F_{R_{\ell j}} - f^0)\frac{v}{\lambda_{Rec_{\ell j}}} \right\} \tag{A-80}$$

and

$$\left(\frac{\partial_e f^1}{\partial t}\right)_{Ion} = -vf^1 \sum_{\ell,j} \frac{1}{\lambda_{IR_{j\ell}}} . \tag{A-81}$$

3. Photoionization and Two-Body Recombination Encounters

On introducing the Lorentz approximation (A-51) into (A-50) we obtain after separation into even and odd powers of μ

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{Ph} = \sum_{\ell,j} \int [f_{n_j} H^{-1} \frac{\bar{I}\beta_{1_{n\ell}}^{n_j}}{\bar{P}_{1_{n\ell}}^{n_j}} - f^0 f_{1_{n\ell}}] g_{1_{n\ell}} \bar{P}_{1_{n\ell}}^{n_j} d^3v_{1_{n\ell}} \tag{A-82}$$

and

$$\left(\frac{\partial_e f^1}{\partial t}\right)_{Ph} = -\int f^1 f_{1_{n\ell}} g_{1_{n\ell}} \bar{P}_{1_{n\ell}}^{n_j} d^3v_{1_{n\ell}} . \tag{A-83}$$

When the small mass approximation is introduced into these equations and the result integrated over the heavy particle velocity there results finally

$$\left(\frac{\partial_e f^0}{\partial t}\right)_{Ph} = \sum_{\ell,j} \left[\frac{n_{n_j}}{n_{1_{n\ell}}} H^{-1} \frac{\bar{I}\beta_{1_{n\ell}}^{n_j}}{\bar{P}_{1_{n\ell}}^{n_j}} - f^0\right]\frac{v}{\lambda_{Ph_{\ell j}}} \tag{A-84}$$

and

$$\left(\frac{\partial_e f^1}{\partial t}\right)_{Ph} = -vf^1 \sum_{\ell,j} \frac{1}{\lambda_{Ph_{\ell j}}} \tag{A-85}$$

where

$$\frac{1}{\lambda_{Ph_{\ell j}}} \equiv n_{1_{n\ell}} \bar{P}_{1_{n\ell}}^{n_j} .$$

For many cases of interest $h\nu_{\ell j} > kT_e$ and Eq. (A-39) can be replaced in the above equations by the approximation

$$\bar{P}_{1_{n\ell}}^{n_j} \approx 4\pi\, \alpha_{1_{n\ell}}^{n_j} . \tag{A-86}$$

Now, if we define F_{NE}, λ_{NE}^0 and λ_{NE} as follows:

$$F_{NE} \equiv \left\{ \sum_{n,k,j} \left(\frac{n_{n_k}}{n_{n_j}} \frac{\omega_j}{\omega_k} \frac{f_0^0}{\lambda_{In_j}} + \frac{n_{n_j}}{n_{n_k}} \frac{\omega_k}{\omega_j} \frac{f_1^0}{\lambda_{In_k}}\right) + \sum_{n,\ell,j} \left(\frac{F_{I_{j\ell}}}{\lambda_{Ion_{j\ell}}} + \frac{F_{R_{\ell j}}}{\lambda_{Rec_{\ell j}}}\right) + \sum_{n,\ell,j} \left(n_{n_j} H^{-1} \bar{I}\beta_{1_{n\ell}}^{n_j}\right) \right\} \lambda_{NE}^0 , \tag{A-87}$$

$$\lambda_{NE}^0 \equiv \sum_{n,k,j} \left(\frac{1}{\lambda_{In_j}} + \frac{1}{\lambda_{In_k}}\right)_n + \sum_{n,\ell,j} \left(\frac{1}{\lambda_{Ion_{j\ell}}} + \frac{1}{\lambda_{Rec_{\ell j}}}\right)_n + \sum_{n,\ell,j} \left(\frac{1}{\lambda_{Ph_{\ell j}}}\right)_n ; \tag{A-88}$$

and

$$\lambda_{NE} = \sum_{n,j,k} \left(\frac{1}{\lambda_{IS_{jk}}}\right)_n + \sum_{n,\ell,j} \left(\frac{1}{\lambda_{IR_{j\ell}}}\right)_n + \sum_{n,\ell,j} \left(\frac{1}{\lambda_{Ph_{\ell j}}}\right)_n , \tag{A-89}$$

then Eqs. (A-60) and (A-61), when combined with Eqs. (A-70, 71, 80, 81, 84, and 85), reduce to Eqs. (A-62) and (A-63) respectively. F_{NE}, which represents the gain of electrons to the set $\vec{v}, d^3v$ from nonelastic encounters, is composed as follows: The first set of terms represent the superelastic and inelastic gain terms of (A-18). The second set of terms in (A-87) correspond to the ionization and recombination gain of (A-34). The remaining set of terms in (A-87) are the gain terms as a result of photoionization. $\lambda_{NE}(v)$ is an effective mean free path for momentum transfer between electrons and heavy particles, resulting from nonelastic encounters. If the nonelastic cross sections are energy dependent only, we have $\lambda_{NE} = \lambda_{NE}^0$. Thus, we can call λ_{NE}^0 an effective mean free path for the nonelastic processes as related to isotropic effects. When $\lambda_{NE} = \lambda_{NE}^0$ only loss (to the set) terms appear in the evaluation of $(\partial_e f^1/\partial t)_{NE}$.

Now, if we define a momentum transfer mean free path λ which includes all the elastic and nonelastic anisotropic collisional contributions to the kinetic equation as

$$\frac{1}{\lambda} = \frac{1}{\lambda_E} + \frac{1}{\lambda_{NE}} \tag{A-90}$$

then (A-62) and (A-63) can be combined with (A-52) and (A-53) to yield

$$\frac{\partial f^0}{\partial t} + \frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{\gamma}{3} v^2 f^1 - \frac{v^4}{2\beta}\frac{\delta}{\lambda_E}\left(\frac{1}{2v}\frac{\partial f^0}{\partial v} + \beta f^0\right)\right.$$

$$\left. - f^0 I_{0,0}^{0,v} - \frac{v}{3}\frac{\partial f^0}{\partial v}\left(I_{2,0}^{0,v} + I_{-1,v}^{0,\infty}\right)\right\} = (F_{NE} - f^0)\frac{v}{\lambda_{NE}^0} \qquad \text{(A-91)}$$

and

$$\frac{\partial f^1}{\partial t} + \gamma\frac{df^0}{\partial v} = -\frac{vf^1}{\lambda} + \left(\frac{\partial_e f^1}{\partial t}\right)_e . \qquad \text{(A-92)}$$

These equations are Eqs. (2) and (3) of Section II.

APPENDIX B

DEVELOPMENT OF A GENERALIZED SPITZER-HÄRM EQUATION

In Section IV-A-2, we found that whenever $X >> \delta$, corresponding to a partially ionized gas, and $Y \leq \delta$ the isotropic part of the distribution function is Maxwellian although at an elevated temperature. That is, f^0 is given by

$$f^0 = \tilde{f}^0 \equiv n_e\left(\frac{\beta_e}{\pi}\right)^{3/2} e^{-\beta_e v^2} . \qquad \text{(B-1)}$$

The anisotropic part of the distribution function is to be determined from Eq. (25) which is rewritten below as (B-2):

$$\gamma\frac{df^0}{dv} = -\frac{vf^1}{\lambda_E} + \left(\frac{\partial_e f^1}{\partial t}\right)_e . \qquad \text{(B-2)}$$

If now we represent f^1 as a multiple of f^0, that is $f^1 = D(v)f^0$, and make use of (B-1) and (A-59), (B-2) can be written as

$$-2\beta_e v\gamma\tilde{f}^0 = -\frac{vD\tilde{f}^0}{\lambda_E} + 8\pi\Gamma_{ee}(\tilde{f}^0)^2 D - \frac{2}{3}\frac{\beta_e\tilde{f}^0}{v}\left(I_{-2,v}^{1,\infty} + I_{1,0}^{1,v}\right)$$

$$+ \frac{4}{5}\beta_e^2 v\tilde{f}^0\left(I_{3,0}^{1,v} + I_{-2,v}^{1,\infty}\right)$$

$$+ \frac{1}{3v}\left[-4\beta_e v\tilde{f}^0\frac{dD}{dv} + f^0\frac{d^2D}{dv^2} + D(4\beta_e^2 v^2\tilde{f}^0 - 2\beta_e\tilde{f}^0)\right]\left(I_{2,0}^{0,v} + I_{-1,v}^{0,\infty}\right)$$

$$+ \frac{1}{3v^2}\left[\tilde{f}^0\frac{dD}{dv} - 2\beta_e\tilde{f}^0 Dv - \frac{\tilde{f}^0 D}{v}\right]\left(3I_{0,0}^{0,v} - I_{2,0}^{0,v} + 2I_{-1,v}^{0,\infty}\right) . \qquad \text{(B-3)}$$

The first two terms in (B-3) correspond respectively to the first two terms in (B-2); the remaining terms in (B-3) correspond to $\left(\frac{\partial_e f^1}{\partial t}\right)_e$. The mean electron energy is included in (B-3) through $\tilde{f}^0$ and is represented explicitly by β_e.

The following identities

$$I^{0,v}_{2,0} + I^{0,\infty}_{-1,v} = \frac{3}{2}\frac{\Gamma_{ee}n_e}{\beta_e v^2}\Lambda \tag{B-4}$$

and

$$3I^{0,v}_{0,0} - I^{0,v}_{2,0} + 2I^{0,\infty}_{-1,v} = 3\Gamma_{ee}n_e - \frac{3}{2}\frac{\Gamma_{ee}n_e}{\beta_e v^2}\Lambda \tag{B-5}$$

where $\Lambda = \phi(\sqrt{\beta_e}\ v) - \sqrt{\beta_e}\ v\ \phi'(\sqrt{\beta_e}\ v)$
and $\phi(\sqrt{\beta_e}\ v) \equiv \mathrm{erf}(\sqrt{\beta_e}\ v)$,
which can be verified after some algebra, can be used to reduce (B-3) to

$$\frac{n_e\Gamma_{ee}\Lambda}{2\beta_e v^3}\frac{d^2D}{dv^2} + \frac{n_e\Gamma_{ee}}{v^2}[\phi-\Lambda(2+\frac{1}{2\beta_e v^2})]\frac{dD}{dv}$$

$$+ [-\frac{v}{\lambda_E} + 8\pi\Gamma_{ee}\tilde{f}^0 + (4\beta_e^2 v + \frac{1}{v^3})(\frac{1}{2}\frac{\Gamma_{ee}n_e}{\beta_e v^2})\Lambda$$

$$- \frac{\Gamma_{ee}n_e}{v^2}(2\beta_e v + \frac{1}{v})\phi]D$$

$$= -2\beta_e v\gamma + \frac{2}{3}\frac{\beta_e}{v}(I^{1,\infty}_{-2,v} + I^{1,v}_{1,0}) - \frac{4}{5}\beta_e^2 v(I^{1,v}_{3,0} + I^{1,\infty}_{-2,v}) . \tag{B-6}$$

The first term on the right-hand side of (B-6) corresponds to the left-hand side of (B-2). The first term in the coefficient of D in (B-6) represents the electron-heavy particle collisions. The isotropic part of the distribution function is related to the error function as follows:

$$\tilde{f}^0 = n_e(\frac{\beta_e}{\pi})^{3/2}\frac{\sqrt{\pi}}{2}\phi' . \tag{B-7}$$

Equation (B-7) can be used in conjunction with (B-6) to reduce the coefficient of D . If (B-6) is multiplied by $2v^3/n_e\Gamma_{ee}\Lambda$ it can be written as:

$$\frac{1}{\beta_e}\frac{d^2D}{dv^2} + [\phi-\Lambda(2+\frac{1}{2\beta_e v^2})]\frac{2v}{\Lambda}\frac{dD}{dv} + \frac{2v^3}{\Lambda}[\frac{\Lambda}{2\beta_e v^5} - (\frac{\phi}{v^3} + \frac{v}{\lambda_E\Gamma_{ee}n_e})$$

$$+ 2\beta_e^{3/2}\phi']D = \frac{2v^3}{n_e\Gamma_{ee}\Lambda}\left\{-2\beta_e v\gamma + \frac{2}{3}\frac{\beta_e}{v}(I^{1,\infty}_{-2,v} + I^{1,v}_{-1,0})\right.$$

$$\left. - \frac{4}{5}\beta_e^2 v(I^{1,v}_{3,0} + I^{1,\infty}_{-2,v})\right\} . \tag{B-8}$$

If we define a term A_s as

$$A_s \equiv \frac{2\gamma}{\beta_e n_e\Gamma_{ee}}$$

which is similar to Spitzer and Härm's "A", the first term on the right-hand side of (B-8) becomes

$$-\frac{2\beta_e^2 v^4 A_s}{\Lambda} .$$

We now introduce some new integrals defined by

$$I_n(x) \equiv \int_0^x y^n D(y)\ e^{-y^2}\ dy . \tag{B-9}$$

These integrals were used by Spitzer and Härm in Ref. 12. An identity can be established between the integrals $I_n(x)$ and I^{1,v_2}_{p,v_1} from which we find:

$$I^{1,v}_{1,0} = \frac{4\Gamma_{ee}n_e}{\sqrt{\pi}\sqrt{\beta_e}v} I_3(\sqrt{\beta_e}v) , \tag{B-10}$$

$$I^{1,v}_{3,0} = \frac{4\Gamma_{ee}n_e}{\sqrt{\pi}\ \beta_e^{3/2}v^3} I_5(\sqrt{\beta_e}v) , \tag{B-11}$$

and

$$I^{1,\infty}_{-2,v} = \frac{4\Gamma_{ee}n_e}{\sqrt{\pi}}\beta_e v^2[I_0(\infty) - I_0(\sqrt{\beta_e}v)] . \tag{B-12}$$

With these expressions we can rewrite (B-8) as

$$\frac{1}{\beta_e}\frac{d^2D}{dv^2} + [\phi-\Lambda(2+\frac{1}{2\beta_e v^2})]\frac{2v}{\Lambda}\frac{dD}{dv} + \frac{2v^3}{\Lambda}[\frac{\Lambda}{2\beta_e v^5} - (\frac{\phi}{v^3} + \frac{v}{\lambda_E\Gamma_{ee}n_e})$$

$$+ 2\beta_e^{3/2}\phi']D = R'(\sqrt{\beta_e}v) + S(\sqrt{\beta_e}v) \tag{B-13}$$

where $$R'(x) \equiv -\frac{2x^4 A_s}{\Lambda} + \frac{16x^4 I_0(\infty)}{3\sqrt{\pi}\,\Lambda}\left(1-\frac{6}{5}x^2\right) , \tag{B-14}$$

and $$S(x) \equiv \frac{16}{3\sqrt{\pi}\,\Lambda}\left\{xI_3(x) - x^4 I_0(x) + \frac{6}{5}x^6 I_0(x) - \frac{6}{5}xI_5(x)\right\} . \tag{B-15}$$

In terms of the variable x, defined as $x \equiv \sqrt{\beta_e}\,v$, (B-13) becomes

$$\frac{d^2D}{dx^2} + P(x)\frac{dD}{dx} + Q'(x)D = R'(x) + S(x) \tag{B-16}$$

with $P(x)$ and $Q'(x)$ defined as follows:

$$P(x) \equiv [\phi - \Lambda(2 + \frac{1}{2x^2})]\frac{2x}{\Lambda} \tag{B-17}$$

$$Q'(x) \equiv \frac{1}{x^2} - \frac{2}{\Lambda}(\phi - 2x^3\phi') - \frac{2x^4}{\beta_e^2 \Lambda n_e \Gamma_{ee} \lambda_E} . \tag{B-18}$$

Equation (B-16) is the same form as the equation solved by Spitzer and Härm and the terms $P(x)$ and $S(x)$ become identical to their $P(x)$ and $S(x)$ when $T_e = T_h$, which is the case for weak electric fields.

The mean free path for elastic electron-heavy particle collisions given by (A-55) can be written as

$$\frac{1}{\lambda_E} = \frac{n_e \Gamma_{ee}}{v^4} + \sum_n \frac{1}{\lambda_n} \tag{B-19}$$

where we have used

$$\sum_i \frac{1}{\lambda_i} = \frac{\Gamma_{ee}}{v^4} \sum_i n_i = \frac{\Gamma_{ee}}{v^4} n_e$$

and the plasma approximation of $\lambda_D >> b_0$ in (A-57). With (B-19) $Q'(x)$ can be written as

$$Q'(x) = \frac{1}{x^2} - \frac{2}{\Lambda}(\phi - 2x^3\phi') - \frac{2x^4}{\beta_e^2 \Lambda n_e \Gamma_{ee}}\left[\frac{n_e \Gamma_{ee} \beta_e^2}{x^4} + \sum_n \frac{1}{\lambda_n}\right]$$

$$= \frac{1}{x^2} - \frac{2}{\Lambda}\left[\phi - 2x^3\phi' + 1 + \frac{x^4}{\beta_e^2 n_e \Gamma_{ee}} \sum_n \frac{1}{\lambda_n}\right] . \tag{B-20}$$

The last two terms in the bracket of (B-20) correspond respectively to the electron-ion and electron-neutral collisions. Equation (B-20) can also be written as

$$Q'(x) = Q(x) - \frac{2}{\Lambda}\frac{\nu_{En}(x)}{\nu_{ee}(x)} . \tag{B-21}$$

For weak fields $[\beta_e = \beta]$ $Q(x)$, defined by

$$Q(x) = \frac{1}{x^2} - \frac{2}{\Lambda}[\phi - 2x^3\phi' + 1] ,$$

is identical to the $Q(x)$ presented by Spitzer and Härm for the case when their mean ionic charge factor equals unity[12]. Equation (B-21) corresponds to (36) for the more general situation of nonequipartition. The total electron-neutral collision frequency $\nu_{En}(x)$ is defined by

$$\nu_{En}(x) \equiv \frac{x}{\sqrt{\beta_e}} \sum_n \frac{1}{\lambda_n} . \tag{B-22}$$

When the plasma is fully ionized $Q'(x) = Q(x)$.

The integral $I_0(\infty)$ appearing in (B-14) is evaluated from the principle of conservation of momentum. If we take the momentum moment of (1), or what is equivalent, multiply (B-2) by $v^3 dv$ and integrate over all v we find

$$\frac{3\gamma\, n_e}{4\pi} = \int \frac{v^4}{\lambda_E} f^1 dv . \tag{B-23}$$

In getting this result we have made use of the fact that electron self-interactions [represented by the last term in (B-2)] cannot change the total momentum of the electrons. If (B-19) and $f^1 = Df^0$ are substituted into (B-23), we obtain

$$\frac{3\gamma\, n_e}{4\pi} = n_e \Gamma_{ee} \int Df^0\, dv + \int v^4 \sum_n \frac{1}{\lambda_n} Df^0\, dv .$$

With (B-1) this last equation can be written as

$$\frac{3\gamma\sqrt{\pi}}{4 n_e \Gamma_{ee} \beta_e} = \int_0^\infty D\, e^{-x^2} dx + \int_0^\infty \frac{\nu_{En}}{\nu_{ee}} e^{-x^2} D dx \tag{B-24}$$

where we have made use of the electron-electron and total electron-neutral collision frequencies defined earlier. The definitions of $I_0(\infty)$ and A_s enable us to write (B-24) as

$$I_0(\infty) = \frac{3\sqrt{\pi}}{8} A_s - \int_0^\infty \frac{\nu_{En}}{\nu_{ee}} e^{-x^2} \, Ddx . \qquad \text{(B-25)}$$

For a fully ionized gas (single ionized) this reduces to Spitzer and Härm's result.

With (B-25) $R'(x)$ can be written as

$$R'(x) = R(x) - \frac{16x^4}{3\sqrt{\pi}\Lambda} (1 - \frac{6}{5} x^2) \int_0^\infty \frac{\nu_{En}(x)}{\nu_{ee}(x)} e^{-x^2} \, Ddx \qquad \text{(B-26)}$$

where $R(x)$ is defined by

$$R(x) \equiv - \frac{2.4x^6 \, A_s}{\Lambda} .$$

For weak electric fields this $R(x)$ is identical to that given by Spitzer and Härm[12] when their mean ionic charge factor equals unity. Equation (B-26) is equivalent to (37) for the more general case of nonequipartition. When the plasma is fully ionized $R'(x) = R(x)$.

Equations (57), (58), and (59), applicable when nonelastic effects are being considered, can be derived for both weak and strong fields in a manner analogous to that outlined above by simply replacing λ_E by λ throughout the analysis.

APPENDIX C
COMPUTATION PROCEDURE

Equations (42, 30, 41, 4, and 6) for f^0 , f^1 , the conductivity, the current density, and the electron temperature along with the energy equation and the Saha equation have been programmed in FORTRAN IV and solved by an iterative technique on the Stanford IBM 7090 digital computer in the region where $X \ll 1$ on Fig. 1. The program evaluates the above quantities based on the plasma constituents, their energy-dependent cross sections, the gas temperature, the total pressure, and the strength of the applied field. The elastic collision cross sections used in the calculations were obtained from Refs. 24 (argon) and 25 (potassium, helium).

The procedure was to first make a consistent calculation of the electron temperature based on a simple energy balance and the Saha equation. In making this calculation the Saha equation was evaluated at a guessed T_e to obtain n_e . This electron number density was used, along with the total pressure and the seed fraction, to determine, via Dalton's law of partial pressures, the number densities of the remaining plasma constituents. These resulting number densities were then used with a Maxwellian distribution function (also evaluated at the guessed T_e) in the energy equation (39) to calculate a new electron temperature. This new temperature replaced the guessed T_e , and the calculation repeated until convergence was obtained in T_e .

Next, to evaluate the exponential in (42) for f^0, we assumed that f^0 was a Maxwellain at T_e where this electron temperature is that obtained in the previous paragraph. We then calculated a first approximation for f^0 and T_e . In this calculation N_0 was determined by normalizing f^0 to the electron number density (obtained via the Saha equation evaluated at the assumed T_e), and T_e was obtained through (6). If the newly calculated values of f^0 and T_e did not agree to within 1% (at every value of the electron speed) and 0.1% respectively of their assumed values, the calculation was repeated with the initially assumed values of f^0 and T_e being replaced by the first approximations. This iterative procedure utilized the Saha equation and Dalton's law of partial pressures for a given seed fraction and total pressure, to determine the species number densities, in conjunction with (42) and (6). The iteration continued until convergence to within the aforementioned criteria in T_e and f^0 was obtained. An energy balance (39) was used to check on the results. Once consistent results were achieved, the desired properties were calculated using Eqs. (30), (4), and (5).

In performing the integrations associated with these computations, Simpson's rule for numerical integration was used with a step size for the electron speed in $(eV)^{1/2}$ of 5×10^{-3}. The integration was performed over the electron speed range of 0 to 5 $(eV)^{1/2}$. Test calculations with a larger step size did not noticeably change the results. Typically for the same constituents, total pressure, and gas temperature, less than a minute of computer time was required to obtain results of f^0, T_e, σ, and j_e for a given field strength.

During the early course of these calculations it was found that whenever f^0 was very close to Maxwellian T_e remained essentially unchanged, within the test criterion of the computer, even for a great number of iterations. For these cases it was necessary to employ an energy balance to obtain the electron temperature*. This peculiarity, which for these cases is a result of the dominance of the electron-electron interaction terms in (42), led to the ordering of the calculation as outlined above. By performing the simple energy balance calculation first we obtained rapid convergence for f^0 and T_e whenever the distribution function was Maxwellian. No convergence difficulties were encountered when f^0 was non-Maxwellian.

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* A similar approach was used in Ref. 18.

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